On the anisotropic velocity distribution of fast ions in NBI-heated toroidal plasmas

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(Dated: 22 March 2018)

A method to calculate the flux-surface-averaged anisotropy (the second Legendre order) in the slowing down velocity distribution of the fast ions generated by tangentially injected neutral beams is shown. This component is required for (1) perpendicular and parallel currents in MHD equilibrium calculations including the fast ions' pressure, (2) the anisotropic heating analyses on the thermalized target plasma species, and (3) the classical and the Pfirsch-Schlüter radial transport of both the thermalized target plasma species and the fast ions themselves. For including the parallel guiding center motion effect in non-symmetric toroidal configurations such as stellarators and heliotrons, the adjoint equation and the eigenfunctions are applied. In contrast to the previously investigated configuration dependence of the first Legendre order as the momentum input to the target plasma species, a quite different dependence of the second Legendre order on the magnetic field strength modulation $B(\theta, \zeta)$ on the magnetic flux-surfaces is found. Even in a low energy range of the slowing down velocity distribution, the deviation (reduction) of the anisotropy from a result neglecting the orbit effect is proportional to $1 - \langle B \rangle / B_{\max}$.

PACS numbers: 52.25.Dg, 52.25.Fi, 52.25.Xz, 52.30.Cv, 52.30.Ex, 52.50.Gj, 52.55.Hc, 52.65.Ff

Keywords: neutral beam injection, anisotropic velocity distribution, neoclassical transport, MHD equilibrium, nonsymmetric toroidal plasmas

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I. INTRODUCTION

In both present experiment devices for the fusion interest and future burning core plasmas in reactors the fast ions play roles as sources of particle, momentum, and energy for thermal particles. In particular, after the development of the charge exchange recombination spectroscopy,[1,2] determination of the velocity distribution of the thermalized ions (H,D,T,He,C, etc.) including this momentum input has been regarded as an important physics issue.[2] This kind of present study also is a step toward the future study on the burning core. By a recent development of the neoclassical theory for general non-symmetric toroidal configurations that is expandable for multi-ion-species plasmas,[3–5] we now can consistently calculate various drift orbits effect in the 3-dimensional real space and collisional non-diagonal couplings between various thermal particles. The inclusion of the parallel momentum input due to the NB (neutral beam)-produced fast ions in this framework was recently conducted and its result successfully explained the experimentally measured impurity flow velocity.[6]

The present study in this paper is motivated by following different contributions of the NB-produced fast ions:
(1) Fast ions' particle flux $n_{\mathrm{f}}\mathbf{u}_{\mathrm{f}} \equiv \int \mathbf{v} f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}$ as a component of the plasma current $\mathbf{J} \equiv \sum_a e_a n_a \mathbf{u}_a$ in the MHD equilibrium is non-negligible not only for the surface-averaged parallel current $\langle \mathbf{B} \cdot \mathbf{J} \rangle$ determining the rotational transform but also for the Pfirsch-Schlüter (P-S) parallel current determining the Shafranov shifts. Here, $\langle \cdot \rangle \equiv \oint \oint \cdot \sqrt{g} \mathrm{d}\theta \mathrm{d}\zeta / \oint \oint \sqrt{g} \mathrm{d}\theta \mathrm{d}\zeta$ is the surface-averaging operation for the poloidal angle θ, and the toroidal angle ζ in appropriately chosen flux-surface coordinates (s, θ, ζ) with the Jacobian $\sqrt{g} \equiv [(\nabla s \times \nabla \theta) \cdot (\nabla \zeta)]^{-1}$. This modification of the shifts by high-energy tangential NBIs is already recognized in various experiments.[7,8] This is one reason for which we investigate tangential NBIs first not only for their flow driving effect but also for their effect on the MHD equilibrium. The anisotropic-pressure MHD equilibrium is defined as a state that includes particle species with the pressure anisotropy being $\left\langle p_{\perp a} - p_{\| a} \right\rangle \left\langle (p_{\perp a} - p_{\| a})/B^2 \right\rangle > 0$. This definition of the "anisotropic-pressure species"is based on the magnetic field curvature effects $\mathbf{b} \cdot \nabla \ln B$ and $\mathbf{b} \cdot \nabla \mathbf{b} \cong \nabla_{\perp} \ln B$ for the unit vector $\mathbf{b} \equiv \mathbf{B}/B$ that are included in the parallel and the perpendicular force balances.[9] In the tangential NBI operations, the P-S current becomes

larger than that in the isotropic-pressure equilibrium with $\sum_a p_a \equiv \sum_a (2p_{\perp a} + p_{\parallel a})/3$ while the perpendicular current decreases. For constructing equilibrium[10,11] and red stability[12–15] theories regarding this situation and for executing such calculations, we should know the pressure moments $p_{\parallel \mathrm{f}}$, $p_{\perp \mathrm{f}}$ of the fast ions' anisotropic gyro-phase-averaged velocity distribution $\overline{f}_{\mathrm{f}}(\mathbf{x}, v, \xi)$ correctly. Hereafter, $\xi \equiv v_{\parallel}/v$ is the cosine of the pitch-angle in the spherical velocity coordinates.

(2) An anisotropic heating for the thermalized target plasma species is another effect of the anisotropic velocity distribution $\overline{f}_{\mathrm{f}}(\mathbf{x}, v, \xi)$. The usual isotropic heating power source term $\int v^2 C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}})\, \mathrm{d}^3\mathbf{v}$ is an energy input in usual energy balance analyses, and the previously investigated first Legendre order of the collision $\left\langle B \int_{-1}^{1} \xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, \overline{f}_{\mathrm{f}}\right) \mathrm{d}\xi \right\rangle$ as an external momentum input term drives the experimentally observable parallel flow moment $\left\langle B \int_{-1}^{1} \xi f_a \mathrm{d}\xi \right\rangle$ that is determined by the balance of the friction collision and the parallel viscosity force. Hereafter, $f_{a\mathrm{M}}(s, v)$ in the Coulomb collision operators $C_{ab}(f_a, f_b)$ or $C_{ba}(f_b, f_a)$ at the minor radial position s in the flux-surface coordinates system is the Maxwellian velocity distribution having the surface-averaged density $\langle n_a \rangle$ of the species a, and the averaged temperatures $\langle T_{\mathrm{e}} \rangle \equiv \langle p_{\mathrm{e}} \rangle / \langle n_{\mathrm{e}} \rangle$ of electrons or $\langle T_{\mathrm{i}} \rangle \equiv \sum_{a \neq \mathrm{e,f}} \langle p_a \rangle / \sum_{a \neq \mathrm{e,f}} \langle n_a \rangle$ of ions. Analogously to this balance, the second Legendre order of the a - f collision $\int_{-1}^{1} P_2(\xi) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, \overline{f}_{\mathrm{f}}\right) \mathrm{d}\xi$ [$P_2(\xi) = \frac{3}{2}\xi^2 - \frac{1}{2}$: Legendre polynomial of order $l = 2$] will generate the surface-averaged pressure anisotropy $\left\langle p_{\perp a} - p_{\parallel a} \right\rangle \left\langle (p_{\perp a} - p_{\parallel a})/B^2 \right\rangle > 0$ of the thermalized target species a by a balance with the anisotropy relaxation collision. The Coulomb collision operator in the neoclassical transport theory is handled based on its characteristic that the test particle portion $C_{ab}(f_a, f_{b\mathrm{M}})$ is a differential operator for the test particles' velocity distribution $f_a(\mathbf{x}, \mathbf{v})$ while the field particle portion $C_{ab}(f_{a\mathrm{M}}, f_b)$ is an integral operator for the field particles' $f_b(\mathbf{x}, \mathbf{v})$. However, the difference between these first and second Legendre orders is not so large. Therefore, the next step issue after investigating the flow driving effect of $\left\langle B \int_{-1}^{1} \xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, \overline{f}_{\mathrm{f}}\right) \mathrm{d}\xi \right\rangle$ is to investigate $\int_{-1}^{1} P_2(\xi) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, \overline{f}_{\mathrm{f}}\right) \mathrm{d}\xi$ since the latter also may generate some experimentally observable changes in the thermalized particles' velocity distributions. However, the ion flow velocities driven by the beam in Ref. 6 were subsonic velocities (i.e., small shifts of the Maxwellian velocity distributions) $\langle n_a \mathbf{u}_a \cdot \mathbf{B} \rangle / \langle n_a B \rangle < 10$km/s caused by the injection velocity of $v_{\mathrm{b}} = 2.28$Mm/s. Furthermore, it is well-known that the beam driven electron flow (so-called shielding current component in the beam driven current) is an order of $u_{\parallel \mathrm{e}} \approx Z_{\mathrm{f}}^2 n_{\mathrm{f}} u_{\parallel \mathrm{f}} / (Z_{\mathrm{eff}} n_{\mathrm{e}})$.[16,17] It also is a

small shift of the electrons' Maxwellian. Because of the aforementioned characteristic of the Coulomb collision, the thermal/fast ratio of the anisotropy $(p_{\mathrm{f}}/p_a)\left|\left(p_{\|a}-p_{\perp a}\right)/\left(p_{\|\mathrm{f}}-p_{\perp \mathrm{f}}\right)\right|$ also will not exceed these $\left|u_{\|a}\right|/v_{\mathrm{b}}$ ratios. This qualitative understanding is one reason for which only the fast ions are regarded as the anisotropic-pressure species, and the thermalized particle species are regarded as the isotropic-pressure species in many previous theories for the MHD equilibrium.[12,13,17] One purpose of this present study is to show a method for confirming the validity of this assumption quantitatively.

(3) In the situation of many present experiments where the $n_{\mathrm{f}}\mathbf{u}_{\mathrm{f}}$ in the $\mathbf{J}\equiv\sum_a e_a n_a \mathbf{u}_a$ (in particular, the P-S parallel current) is non-negligible, the P-S and the classical radial particle/energy diffusions (defined in Ref.9) of thermalized particles caused by the friction (momentum exchange) collision also are modified. This transport process will be important especially for impurities. Simultaneously, this friction causes also the classical and the P-S diffusions of fast ions themselves. In past calculations of the $\overline{f}_{\mathrm{f}}(\mathbf{x},v,\xi)$, this kind of radial transport is often regarded as a higher order of ρ_{f}/L_r [ρ_a : typical circulating orbit deviation of the species a from the flux-surfaces, $L_r\equiv\left|\nabla_{\perp}\ln\sum_a p_a\right|^{-1}$: radial gradient scale length] and is neglected. To investigate the radial transport of fast ions themselves is important as a confirmation of the validity of these calculations of the $\overline{f}_{\mathrm{f}}(\mathbf{x},v,\xi)$. Analogous to the calculation of the P-S parallel and the perpendicular currents, this friction calculation also requires knowledge regarding the anisotropy (the second Legendre order).

The purpose of this study is to show a method to calculate the flux-surface-averaged anisotropy in the $\overline{f}_{\mathrm{f}}(\mathbf{x},v,\xi)$ of the NB-produced fast ions for these applications. However, this velocity distribution in non-symmetric toroidal configurations will have a complicated phase space structure because the non-uniform magnetic field strength $\mathbf{B}\cdot\nabla B\neq 0$ in the 3D real space makes three types of phase space regions, i.e., circulating, toroidally trapped, and ripple-trapped regions corresponding to different drift orbits. In the drift kinetic equation (DKE) for the fast ions, a set of $\sigma\equiv v_{\|}/\left|v_{\|}\right|=\pm 1$ and $\lambda\equiv\mu B_{\mathrm{M}}/w\equiv(B_{\mathrm{M}}/B)v_{\perp}^{2}/v^{2}$ with the maximum magnetic field strength B_{M} on each flux-surface is used mainly as the pitch-angle space parameter, rather than $\xi\equiv v_{\|}/v=\sigma\left(1-\lambda B/B_{\mathrm{M}}\right)^{1/2}$. The circulating and the trapped pitch-angle ranges are defined as $0\leq\lambda\leq 1$ and $1<\lambda\leq B_{\mathrm{M}}/B$, respectively. The latter region consists of the ripple-trapped range $0\leq\kappa^2\leq 1$ and the toroidally trapped range $\kappa^2>1$ where κ^2 is defined by $\kappa^2\equiv\left\{\left(B_{\mathrm{M}}/B_0\right)/\lambda-\left(1+\varepsilon_{\mathrm{T}}-\varepsilon_{\mathrm{H}}\right)\right\}/(2\varepsilon_{\mathrm{H}})$ for stellarator/heliotron magnetic fields $B/B_0 = 1 + \varepsilon_{\mathrm{T}}(s,\theta) + \varepsilon_{\mathrm{H}}(s,\theta)\cos[L\theta - N\zeta + \gamma(s,\theta)]$ with the volume averaged field strength B_0. In both numerical (see references cited in Ref.3) and analytical[18] methods for solving these kinds of kinetic equations, appropriate calculation methods for the drift orbit and the collision effects must be chosen and used complimentarily for these pitch-angle ranges, if we need the complete determination of the solution $f_a(\mathbf{x},\mathbf{v})$ in the full phase space regions. From the viewpoint of various effects of the fast ions on the thermalized target plasma particle species such as that caused by the Coulomb collision or the direct contribution of $n_{\mathrm{f}}\mathbf{u}_{\mathrm{f}}$ as the component of the current in the MHD equilibrium, however, we do not need to know the complete structure of $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$ itself. For its pitch- and gyro-angle space structure that is handled by the spherical harmonic expansion, the Coulomb collision operator (field particle portion[19]) $C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}})$ is an integral operator suppressing the higher Legendre orders, and we should investigate the aforementioned lower Legendre orders $l = 0, 1, 2$ (corresponding to the energy input, the momentum input, and the anisotropic heating, respectively) first. For the MHD equilibrium, we need only to know $p_{\|\mathrm{f}} + p_{\perp\mathrm{f}}$, $(p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B^2$ for the perpendicular and P-S parallel current, and $\left\langle B n_{\mathrm{f}} u_{\|\mathrm{f}} \right\rangle$ for the beam driven current. On the real space structure of $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$ in this MHD calculation, it should be emphasized that the construction of the aforementioned flux-surface coordinates system for the transport analyses is included in the purpose of this calculation. This coordinates system is constructed for the $\mathbf{B}, \mathbf{J}$ vector fields basically satisfying $\nabla\cdot\mathbf{B} = \nabla\cdot\mathbf{J} = \mathbf{B}\cdot\nabla s = \mathbf{J}\cdot\nabla s = 0$. Therefore only the surface-averaged contributions $\left\langle p_{\|\mathrm{f}} + p_{\perp\mathrm{f}} \right\rangle$ and $\left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B^2 \right\rangle$ of the fast ions' pressure are required[9] even if the solution of the DKE for $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$ itself may have complicated phase space structures. This is a reason why we adopt the adjoint equation method[20] for the $\left\langle \int x_a^2 P_2(\xi) L_j^{(5/2)}(x_a^2) C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}})\, \mathrm{d}^3\mathbf{v}/B \right\rangle$ integrals with $j = 0, 1, 2$ of the Coulomb collision operator (Sec.III) $[L_j^{(\alpha)}(K) \equiv \left(e^K K^{-\alpha}/j!\right) \mathrm{d}^j \left(e^{-K} K^{j+\alpha}\right)/\mathrm{d}K^j$ and $x_a \equiv v/\sqrt{2\langle T_a \rangle/m_a} \equiv v/v_{\mathrm{T}a}]$ and the $\left\langle \int v^k P_2(\xi) f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}/B \right\rangle$ integrals with $k = -1, 1, 2, 4, 6$ of the velocity distribution function (Sec.IV) in this study, instead of direct solution methods for the $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$ itself such as the previously proposed eigenfunction that is defined for the full pitch-angle range $0 \leq \lambda \leq B_{\mathrm{M}}/B$.[21]

It was found in a previous study[9] that the friction moments of the collision between thermal and fast ions $\left\langle B \int v\xi L_j^{(3/2)}(x_a^2) C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}})\, \mathrm{d}^3\mathbf{v} \right\rangle$ in the tangential NBI operations have a strong dependence on the $\mathbf{B}$-field strength modulation on the flux surfaces

$\left\langle (1-B/B_{\rm M})^{1/2} \right\rangle \neq 0$. This is due to the fast ions' parallel (to $\mathbf{B}$) guiding center motion conserving the magnetic moment μ. Analogous to the banana regime parallel viscosity of thermalized particles[22,23], a connection of independently determined solutions for $\sigma = \pm 1$ at $v_\parallel = 0$ is used also in the solving procedures of the DKE for the fast ions. Therefore the previously found dependence of the velocity distribution function $\overline{f}_{\rm f}^{(\rm odd)} \equiv \left[\overline{f}_{\rm f}(\mathbf{x}, v, \sigma, \lambda) - \overline{f}_{\rm f}(\mathbf{x}, v, -\sigma, \lambda)\right]/2$ on the field strength modulation will appear also in $\overline{f}_{\rm f}^{(\rm even)} \equiv \left[\overline{f}_{\rm f}(\mathbf{x}, v, \sigma, \lambda) + \overline{f}_{\rm f}(\mathbf{x}, v, -\sigma, \lambda)\right]/2$. Only one difference from the $\overline{f}_{\rm f}^{(\rm odd)}(\mathbf{x}, v, \sigma, \lambda)$ is that the trapped pitch-angle range $1 < \lambda \leq B_{\rm M}/B$ must be included when we calculate the anisotropy $\int_{-1}^{1} P_2(\xi)\overline{f}_{\rm f}{\rm d}\xi$. In this previous investigation of $\overline{f}_{\rm f}^{(\rm odd)}$, we used a direct solving method for the fast ion DKE. This method was possible since this component exists only in the circulating pitch-angle range $0 \leq \lambda \leq 1$. The mathematical method (eigenfunction) originally developed for axisymmetric tokamaks[24] was easily generalized to non-symmetric stellarator/heliotron configurations. On the other hand, the $\int_{-1}^{1} P_2(\xi)\overline{f}_{\rm f}{\rm d}\xi$ integral as the purpose of this present study requires $\overline{f}_{\rm f}^{(\rm even)}$ that can be broadened to the full pitch-angle range $0 \leq \lambda \leq B_{\rm M}/B$ by the pitch-angle scattering (PAS) collision during the slowing down process. Cordey proposed a method for expressing this velocity distribution function using the eigenfunctions that are defined for the full pitch-angle range.[21] However, numerous analytical approximations assuming axisymmetric tokamaks that can be justified only for $\left\langle (1-B/B_{\rm M})^{1/2} \right\rangle \ll 1$ were used. This method cannot be generalized to non-symmetric stellarator/heliotron configurations. Instead of that, we shall adopt the adjoint equation method that was previously used by Taguchi for calculating the fast ions' parallel particle flux $\langle B n_{\rm f} u_{\parallel \rm f} \rangle$,[20] since our purpose is not in the $\overline{f}_{\rm f}^{(\rm even)}(\mathbf{x}, v, \lambda)$ itself but in some surface-averaged contributions of the velocity space integrals $\int {\rm d}^3\mathbf{v}$ of this function. Even though this adjoint equation also is defined for the full phase space regions $(\mathbf{x}, \mathbf{v})$, we need its solution only at a specific pitch-angle range where the fast ion source exists. For the tangential NBIs, the required solution is that in the circulating pitch-angle range $0 \leq \lambda \leq 1$, and thus this method can be commonly used for general toroidal configurations (not only axisymmetric tokamaks but also non-symmetric stellarator/heliotron devices).

Therefore, the rest of this work is organized as follows. In Sec.II, the DKE for the fast ions in the tangential NBI operations and the adjoint equation method are introduced. Based on them, a formula that is applicable for various integrals in above applications with a common form $\left\langle \int H_2(v)P_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B \right\rangle$ is derived. The application of this formula to the anisotropic heating analysis is shown in Sec.III. However, an important issue in this section is a relation of this newly added part in the thermalized particles' DKE with the previously studied parts for handling various radial gradient forces and parallel forces. As responses to these forces, the poloidally and toroidally varying anisotropies $p_{\|a} - p_{\perp a}$ and $r_{\|a} - r_{\perp a}$ are generated corresponding to the neoclassical viscosities that also have the CGL (Chew-Goldberger-Low) form $\boldsymbol{\pi}_a = (p_{\|a} - p_{\perp a})\,(\mathbf{bb} - \mathbf{I}/3)$, $\mathbf{r}_a - r_a\mathbf{I} = (r_{\|a} - r_{\perp a})\,(\mathbf{bb} - \mathbf{I}/3)$ with the unit tensor $\mathbf{I}$. The generation of these velocity distribution components is complicated rather for the thermalized particles than for the fast ions in Sec.II, since the thermalized particles' velocity distribution is strongly affected by the field particle portion $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}\right)$ in the linearized collision operator as a coupling between DKEs for different particle species, and the $\mathbf{E}\times\mathbf{B}$ drift due to the ambipolar radial electric field $-\partial\Phi/\partial s$. The collision and the drift approximations must be appropriately chosen for these components. This difference between the fast ions and the thermalized particles will be discussed in Sec.III. In Sec.IV, methods for calculating the parallel and the perpendicular flow moments of the fast ions and various fiction integrals caused by them are shown for the P-S and the classical diffusions of both thermalized particles and the fast ions themselves. The $n_{\mathrm{f}}\mathbf{u}_{\mathrm{f}}$ in the current caused by the fast ions' anisotropic pressure also will be obtained by a method in this section. The summary is given in Sec.V. In these discussions, the knowledge regarding the perpendicular and the parallel particle/energy fluxes (in particular, that on the solubility condition of the parallel fluxes) of general particle species with non-negligible anisotropies is required. This explanation is given in Appendix A. The anisotropic heating analysis in Sec.III utilizes the Braginskii's matrix expression of the anisotropy relaxation collision based on the three terms Laguerre expansion. The required matrix elements are summarized in Appendix B.

II. ADJOINT EQUATION FOR INCLUDING FAST IONS' PARALLEL GUIDING CENTER MOTION EFFECTS

The drift kinetic equation for the tangentially injected fast ions is given by[9,20,21]

$$\begin{aligned} V_\parallel \overline{f}_{\mathrm{f}} &= \sum_b C_{\mathrm{f}b}\left(\overline{f}_{\mathrm{f}}, f_b\right) + S_{\mathrm{f}}(s,v,\sigma,\lambda), \\ V_\parallel \equiv v_\parallel \mathbf{b}\cdot\nabla_{(v,\lambda)=\mathrm{const}} &= v\xi\mathbf{b}\cdot\nabla_{(v,\xi)=\mathrm{const}} - \frac{v}{2}\frac{\mathbf{b}\cdot\nabla B}{B}(1-\xi^2)\frac{\partial}{\partial\xi}. \end{aligned} \tag{1}$$

In this section, we shall investigate the 0th order of $\rho_{\mathrm{f}}/L_r \propto \langle B\rangle^{-1}$ in the gyro-phase-averaged velocity distribution $\overline{f}_{\mathrm{f}}(\mathbf{x},v,\sigma,\lambda)$ by using this equation excluding the perpendicular guiding center drift velocity $\mathbf{v}_{\mathrm{df}} = (c/e_{\mathrm{f}})\left(m_{\mathrm{f}}v_\parallel^2/B + \mu\right)\mathbf{b}\times\nabla\ln B$. Although we also will investigate the poloidal/toroidal variations of the 1st Legendre order moment $\int_{-1}^{1}\xi\overline{f}_{\mathrm{f1}}\mathrm{d}\xi$ (contributing to the P-S current and causing the P-S radial transport) as a response to the radial gradient term $(\mathbf{v}_{\mathrm{df}}\cdot\nabla s)\,\partial\overline{f}_{\mathrm{f}}^{(\mathrm{even})}/\partial s$ for $\overline{f}_{\mathrm{f}}^{(\mathrm{even})} \equiv \left[\overline{f}_{\mathrm{f}}(\mathbf{x},v,\sigma,\lambda) + \overline{f}_{\mathrm{f}}(\mathbf{x},v,-\sigma,\lambda)\right]/2$ after this determination of the 0th order component (see Sec.IV), this $\overline{f}_{\mathrm{f1}}$ is the 1st order of ρ_{f}/L_r (a velocity distribution function component being $\propto \langle B\rangle^{-1}$). Note that the perpendicular differential $\nabla_\perp f_a$ in the DKEs in this paper is that keeping constant (v,ξ), not only in the $\mathbf{E}\times\mathbf{B}$ operator V_E introduced below but also in this $\mathbf{v}_{\mathrm{d}a}\cdot\nabla f_a$. Since $\nabla B\times\mathbf{B}\cdot\nabla B = 0$, the field curvature effect $\mathbf{b}\cdot\nabla\mathbf{b} \cong \nabla_\perp \ln B$ does not appear in the $\nabla_\perp f_a$ for the purpose of the $\mathbf{v}_{\mathrm{d}a}\cdot\nabla f_a$. This is in contrast to the gyro-phase-dependent part in Sec.IV and the other curvature effect $\mathbf{b}\cdot\nabla\ln B$ in $V_\parallel$. This procedure, in which $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial f_a/\partial s$ is added after obtaining the solution being $V_\parallel f_a = 0$ of the equation excluding $\mathbf{v}_{\mathrm{d}a}\cdot\nabla f_a$, is analogous to that for transport of fusion-born fast ions in burning plasmas.[24] The procedure will be applied also for the fast-ion-driven component of $\left\langle B^{-1}\int_{-1}^{1}P_2(\xi)f_a\mathrm{d}\xi\right\rangle$ $(a\neq \mathrm{f})$ in Sec.III if it is not negligibly small. It also is known regarding this drift velocity $\mathbf{v}_{\mathrm{df}}$ that the poloidal precession of the deeply trapped particles in $\kappa^2 \le 1$ due to $\partial\varepsilon_{\mathrm{H}}/\partial s$ and $\partial\varepsilon_{\mathrm{T}}/\partial s$ is sometimes important together with the radial component $\mathbf{v}_{\mathrm{df}}\cdot\nabla s$.[25] In this study, however, we shall assume that the fast ion source term $S_{\mathrm{f}}(s,v,\sigma,\lambda) \propto \delta(v-v_b)/v^2$ (given by HFREYA and MCNBI codes[26] that are used and/or assumed in recent related works[6,9]) exists only in the circulating pitch-angle range $\lambda < 1$ (i.e., tangential NBI). In this case, the trapped particles in $\kappa^2 \le 1$ that will be generated by the PAS collision during the slowing down process exist only in the low-energy region $v < v_{\mathrm{c}}$. Here, v_{c} is the critical velocity that will be introduced in the fast ion collision operator Eq.(2). Therefore, to allow the $-c\nabla\Phi\times\mathbf{B}/B^2$ precession and the collisionless detrapping/retrapping of the low-energy trapped particles in $\kappa^2 \leq 1$ caused by the ambipolar electrostatic potential[27] in following discussions is more important than the radial gradients $\partial\varepsilon_{\mathrm{H}}/\partial s$, $\partial\varepsilon_{\mathrm{T}}/\partial s$ of the $\mathbf{B}$-field strength.

In spite of this implicit allowing of $-c\nabla\Phi\times\mathbf{B}/B^2$ for $v<v_{\mathrm{c}}$ and $\kappa^2\leq 1$ in the adjoint equation method, the electric field term $\mathbf{E}\cdot\partial\overline{f}_{\mathrm{f}}/\partial\mathbf{v}$ in the Vlasov operator is not taken into account explicitly at least in Eq.(1) analogously to the previous investigation of the parallel momentum input. In particular, the $\mathbf{E}\times\mathbf{B}$ flow divergences in Eqs.(A14-A15) caused by the ambipolar electrostatic potential being an order of $|\nabla\Phi|\sim|(\nabla p_a)/(e_a n_a)|$ $(a\neq \mathrm{f})$ are not important for high-energy ions with $m_{\mathrm{f}}v_{\mathrm{b}}^2/2\gg T_{\mathrm{e}},T_{\mathrm{i}}$ while the divergences correspond to a part of the thermodynamic force in the DKEs for thermalized particles.[3–6] The various $\int \mathbf{v}v^{k-2}f_a\mathrm{d}^3\mathbf{v}$ flow moments (in Sec.IV) of various particle species caused by the $-c\nabla\Phi\times\mathbf{B}/B^2$ drift vanish in the perpendicular and the P-S parallel current, and the classical and the P-S radial transport because of the charge neutrality and the Galilean invariant property of the Coulomb collision. For the inductive parallel electric field $\left\langle\mathbf{B}\cdot\mathbf{E}^{(\mathbf{A})}\right\rangle\mathbf{B}/\left\langle B^2\right\rangle$, the momentum conservation of electron-ion and ion-electron collisions is the reason for this omission of $E_{\parallel}^{(\mathbf{A})}\partial\overline{f}_{\mathrm{f}}/\partial v_{\parallel}$. When the Joule heating current is generated by $E_{\parallel}^{(\mathbf{A})}$, the conservation law proves that $m_a\int v\xi C_{a\mathrm{e}}\left(\frac{1}{2}\int_{-1}^{1}\overline{f}_a\mathrm{d}\xi, f_{\mathrm{e}}\right)\mathrm{d}^3\mathbf{v}\approx -\left(Z_a/Z_{\mathrm{eff}}\right)e_a n_a E_{\parallel}^{(\mathbf{A})}$ for both thermalized and fast ions $a\neq \mathrm{e}$. Here, the so-called effective charge number $Z_{\mathrm{eff}}\equiv\sum_{a\neq \mathrm{e,f}} Z_a^2 n_a/n_{\mathrm{e}}$ is assumed to be order unity. In the DKEs for thermalized ions $a\neq$ e,f, to retain $E_{\parallel}^{(\mathbf{A})}\partial\overline{f}_{\mathrm{f}}/\partial v_{\parallel}$ for the confirmation of the Onsager symmetric relation between the bootstrap current and the Ware pinch is meaningful when the collision term $C_{a\mathrm{e}}\left(f_{a\mathrm{M}},f_{\mathrm{e}1}\right)\cong m_{\mathrm{e}}\mathbf{v}\cdot\left\{\int\mathbf{v}\nu_{\mathrm{D}}^{\mathrm{e}a}(v)f_{\mathrm{e}}\mathrm{d}^3\mathbf{v}\right\}f_{a\mathrm{M}}/p_a$ for $f_{\mathrm{e}}=f_{\mathrm{eM}}+f_{\mathrm{e}1}$ given by the usual small mass ratio approximation for the ion-electron collision is simultaneously retained. On the other hand, in the Eq.(1) for the fast ions, to retain the inductive field term is meaningless as long as we use the collision approximation $C_{\mathrm{f}a}\left(f_{\mathrm{f}},f_a\right)\cong C_{\mathrm{f}a}\left(f_{\mathrm{f}},f_{a\mathrm{M}}\right)$ for $a\neq \mathrm{f}$ in which field particles' flow velocity moments $\mathbf{u}_a\equiv\int\mathbf{v}f_a\mathrm{d}^3\mathbf{v}/n_a$ in their shifted Maxwellian distributions $f_{a\mathrm{M}}\left(\mathbf{v}-\mathbf{u}_a\right)$ generated by various mechanisms are neglected. The reason for this collision approximation is that the flows of the thermalized target plasma species are often subsonic flows (i.e., the electron flow is $|\mathbf{u}_{\mathrm{e}}|\ll v_{\mathrm{b}}$ and that of thermalized ions is $|\mathbf{u}_a|\ll v_{\mathrm{c}}$) in various experimental measurements and theoretical calculations.

This fast ion collision operator is given by[9]

$$
\begin{aligned}
\sum_a C_{\mathrm{fa}}\left(f_{\mathrm{f}}, f_a\right) &\cong \sum_{a\neq \mathrm{f}} C_{\mathrm{fa}}\left(f_{\mathrm{f}}, f_{a\mathrm{M}}\right) \\
&\cong \frac{1}{\tau_{\mathrm{S}}}\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2 v_{\mathrm{Te}}\frac{3\sqrt{\pi}}{2}G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3\right\} f_{\mathrm{f}}+\frac{Z_2 v_{\mathrm{c}}^3}{v^3}\mathcal{L} f_{\mathrm{f}}\right] \equiv C_{\mathrm{f}} f_{\mathrm{f}}, \\
\mathcal{L} &\equiv \frac{1}{2}\left(\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}+\frac{1}{1-\xi^2}\frac{\partial^2}{\partial\phi^2}\right) \\
&= \frac{B_{\mathrm{M}}}{B}\left(2\frac{v_\parallel}{v}\frac{\partial}{\partial\lambda}\lambda\frac{v_\parallel}{v}\frac{\partial}{\partial\lambda}+\frac{1}{2\lambda}\frac{\partial^2}{\partial\phi^2}\right).
\end{aligned} \tag{2}
$$

In contrast to the collision operators for the thermalized particles discussed in Sec.III, the nonlinear collision term $C_{\mathrm{ff}}\left(f_{\mathrm{f}}, f_{\mathrm{f}}\right)$ is omitted because of the low density of fast ions themselves and the momentum/energy conservation of like-particle collisions. In addition to the neglect of the shifts $\mathbf{u}_a$ of the Maxwellians and consequently of the Rosenbluth potentials[19] $\mathcal{H}\left(f_a\right)$, $\mathcal{G}\left(f_a\right)$ for $a \neq$ f, higher Legendre orders $l \geq 2$ in the potentials also are neglected because of their characteristic as integral operators and a nearly thermalized state $|f_{a1}| \ll f_{a\mathrm{M}}$ for $f_a\left(\mathbf{x},\mathbf{v}\right) = f_{a\mathrm{M}}\left(s, v\right)+f_{a1}\left(\mathbf{x},\mathbf{v}\right)$ of the thermalized target particles species at the thermal energy range $m_a v^2 \sim 2T_a$. This relation will be confirmed in Sec.III. The constants τ_{S}, Z_2, and v_{c}^3 being independent of $(\theta, \zeta, \mathbf{v})$ on each flux-surfaces are defined in Ref.9. The PAS parameter Z_2 is a dimensionless coefficient of order unity ($Z_2 \approx Z_{\mathrm{eff}} \geq 1$ for NB-produced fast ions), and the critical velocity v_{c} has a relation $\left(v_{\mathrm{c}}/v_{\mathrm{Te}}\right)^3 \approx (3\sqrt{\pi}/8)m_{\mathrm{e}}/m_{\mathrm{amu}} = 3.65\times 10^{-4}$ with the electron thermal velocity where $m_{\mathrm{amu}} = 1.66\times 10^{-27}$kg is the atomic mass unit (dalton). This approximation of the PAS term $\propto \mathcal{L}$ in this study where $Z_2 v_{\mathrm{c}}^3/\tau_{\mathrm{S}}$ is handled as a constant in the full velocity range $0 \leq v \leq v_{\mathrm{b}}$ is justified by a fact indicated by the resultant solution that the PAS collision for the second Legendre order $\left\langle \int_{-1}^{1} P_2(\xi) f_{\mathrm{f}} \mathrm{d}\xi/B\right\rangle$ is substantially effective only in a slow velocity range $v \lesssim v_{\mathrm{c}}$. The Chandrasekhar function $G\left(x\right) \equiv \frac{2}{\sqrt{\pi}}x^{-2}\int_0^x y^2 \exp\left(-y^2\right)\mathrm{d}y$ with $x_{\mathrm{e}} \equiv v/v_{\mathrm{Te}}$, which can be easily calculated by using the usual error function, is included corresponding to the slowing down collision frequency of the f-e collision. Although this slowing down term includes an artificial violation of the particle conservation $\int C_{\mathrm{fa}}\left(f_{\mathrm{f}}, f_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v} = 0$ of collisions of the fast and the thermalized ions ($a \neq$ e,f), this violation concerns only the non-zero boundary value $f_{\mathrm{f}}\left(v=0\right) \neq 0$ of the lowest Legendre order component $f_{\mathrm{f}}^{(l=0)} \equiv \frac{1}{2}\int_{-1}^{1}\overline{f}_{\mathrm{f}}\mathrm{d}\xi$ that determines the particle fueling to the thermalized ion species with $m_a = m_{\mathrm{f}}$, $e_a = e_{\mathrm{f}}$. This fueling effect is irrelative to the generation of $\mathbf{B}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi/B\right)$ in the suprathermal energy range that will be discussed in this section and Sec.IV.

In this study for some velocity space integrals with the common form of $\int P_2(\xi)H_2(v)\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}$, the aforementioned $\overline{f}_{\mathrm{f}}^{(\mathrm{even})}(\mathbf{x},v,\lambda)$ as an even function of $v_\parallel$ defined for the full pitch-angle range $0\leq\lambda\leq B_{\mathrm{M}}/B$ is required. Although it may be possible to obtain the expression of this function if the configuration is a simple axisymmetric tokamak. In fact, Cordey previously proposed an expressing method using eigenfunctions that are defined for the full range.[21] The eigenfunction for expressing arbitrary functions in $0\leq\lambda\leq1$ was defined by using the surface-averaged parallel particle velocity $\left\langle(1-\lambda B/B_{\mathrm{M}})^{1/2}\right\rangle$ as shown in Refs.9 and 24. Hereafter, the surface-averaging operation $\langle F(\lambda)\rangle$ for functions of λ or ξ is the average keeping constant λ values. In the Cordey's eigenfunction, this concept is extended to the trapped pitch-angle range $1<\lambda\leq B_{\mathrm{M}}/B$ by using the bounce-integrals. However, this method cannot be easily generalized to non-symmetric stellarator/heliotron configurations since there are two types of the pitch-angle space regions $0\leq\kappa^2\leq1$ and $\kappa^2>1$. Since our purpose is to obtain some surface-averaged contributions of the $\int P_2(\xi)H_2(v)\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}$ integrals as stated in the introduction, instead of this kind of direct solving method for Eq.(1), we shall adopt the following adjoint equation method.[20] The adjoint equation has the following form:

$$\begin{aligned}&\left(V_\parallel+C_{\mathrm{f}}^{\mathrm{A}}\right)f_{\mathrm{A}}=\sigma_{\mathrm{A}}(\mathbf{x},v,\sigma,\lambda),\\ C_{\mathrm{f}}^{\mathrm{A}}&\equiv\frac{1}{\tau_{\mathrm{S}}}\left[-\frac{v^2v_{\mathrm{Te}}(3\sqrt{\pi}/2)G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3}{v^2}\frac{\partial}{\partial v}+\frac{Z_2v_{\mathrm{c}}^3}{v^3}\mathcal{L}\right].\end{aligned}\tag{3}$$

This differential operator $C_{\mathrm{f}}^{\mathrm{A}}$ satisfies

$$\int H\left(C_{\mathrm{f}}^{\mathrm{A}}F\right)\mathrm{d}^3\mathbf{v}=\int F\left(C_{\mathrm{f}}H\right)\mathrm{d}^3\mathbf{v}\tag{4}$$

for arbitrary functions satisfying $F\left(v=0\right)=0$ and $H\left(v=\infty\right)=0$. Following this relation and the anti-symmetric property $\left\langle\int_{-1}^{1}F\left(V_\parallel H\right)\mathrm{d}\xi\right\rangle=-\left\langle\int_{-1}^{1}H\left(V_\parallel F\right)\mathrm{d}\xi\right\rangle$ of the parallel orbit propagator $V_\parallel$, we can obtain a relation

$$\begin{aligned}&\left\langle\int\sigma_{\mathrm{A}}\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\right\rangle=\left\langle\int\overline{f}_{\mathrm{f}}\left(V_\parallel f_{\mathrm{A}}\right)\mathrm{d}^3\mathbf{v}\right\rangle+\left\langle\int\overline{f}_{\mathrm{f}}\left(C_{\mathrm{f}}^{\mathrm{A}}f_{\mathrm{A}}\right)\mathrm{d}^3\mathbf{v}\right\rangle\\ &=-\left\langle\int f_{\mathrm{A}}\left(V_\parallel\overline{f}_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v}\right\rangle+\left\langle\int f_{\mathrm{A}}\left(C_{\mathrm{f}}\overline{f}_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v}\right\rangle=-\left\langle\int S_{\mathrm{f}}(s,v,\sigma,\lambda)f_{\mathrm{A}}\mathrm{d}^3\mathbf{v}\right\rangle.\end{aligned}\tag{5}$$

When this method is applied to the tangential NBIs where the fast ion source $S_{\mathrm{f}}(s,v,\sigma,\lambda)$ exists only in $\lambda<1$, we can immediately know $\left\langle\int\sigma_{\mathrm{A}}\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\right\rangle$ only by solving Eq.(3) for f_{A}

in $\lambda < 1$ without solving Eq.(1) for $\overline{f}_{\mathrm{f}}$ in the full pitch-angle range. We shall calculate the $\left\langle \int x_a^2 P_2(\xi) L_j^{(5/2)}(x_a^2) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right) \mathrm{d}^3\mathbf{v}/B \right\rangle$ integrals of the Coulomb collision operator (for the anisotropic heating analyses in Sec.III) and the $\left\langle \int v^k P_2(\xi) f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}/B \right\rangle$ integrals of the velocity distribution function (for the current, and the P-S and the classical radial transport in Sec.IV) in general toroidal configurations by this method. As noted on Eq.(1), this adjoint equation method implicitly allows the poloidal precession of the deeply trapped particles in $\kappa^2 \leq 1$ and their collisionless detrapping/retrapping by various mechanisms since the relation Eq.(5) holds even when the operator $V_\parallel$ is replaced by $V_\parallel + V_E$ that is often used for the thermalized particles. Here, V_E is defined by[27]

$$V_E \equiv cE_s \frac{\nabla s \times \mathbf{B}}{B^2} \cdot \nabla_{(v,\xi)=\mathrm{const}} + cE_s \left(\frac{\nabla s \times \mathbf{B}}{2B^2} \cdot \nabla \ln B \right) \left\{ (1-\xi^2)\xi \frac{\partial}{\partial \xi} + (1+\xi^2) v \frac{\partial}{\partial v} \right\} \quad (6)$$

with $E_s \equiv -\partial\Phi/\partial s$ being constant on each flux-surface, and satisfies $\left\langle \int H \left(V_E F\right) \mathrm{d}^3\mathbf{v} \right\rangle = -\left\langle \int F \left(V_E H\right) \mathrm{d}^3\mathbf{v} \right\rangle$. The basic idea of the fast/thermal separation[9] that the $f_{\mathrm{f}}\left(\mathbf{x}, \mathbf{v}\right)$ determined by the collision operator in Eq.(2) will have the isotropic pitch-angle space structure and the flat energy space structure at the low energy limit $v^3 \ll v_{\mathrm{c}}^3$ also due to this actual existence of the $-c\nabla\Phi \times \mathbf{B}/B^2$ drift that eliminates the velocity space loss region.[25] The statement on Eq.(1) that we do not calculate $\partial\varepsilon_{\mathrm{H}}/\partial s$, $\partial\varepsilon_{\mathrm{T}}/\partial s$, and $\partial\Phi/\partial s$ means an approximation neglecting these quantities only for the solution of Eq.(3) in $\lambda < 1$ analogously to the theory of thermal particles' banana regime parallel viscosity in general toroidal plasmas.[22,23]

Although this method in Eqs.(3)-(5) can be applied for arbitrary function $\sigma_{\mathrm{A}}(\mathbf{x}, v, \sigma, \lambda)$ as long as the boundary condition at $v = 0, \infty$ in the energy space is satisfied, we investigate only cases of $\sigma_{\mathrm{A}}(\mathbf{x}, v, \sigma, \lambda) \propto P_2(\xi)/B\left(\theta, \zeta\right)$ in this paper because of the following two reasons. The first reason is the DKE for thermalized particles that will be shown in Sec.III. The additional velocity distribution component caused by the anisotropic heating effect has a form of $\propto P_2(\xi)/B - 1/B + \langle 1/B \rangle$ that satisfies $V_\parallel\left(P_2(\xi)/B - 1/B + \langle 1/B \rangle\right) = 0$. The determination of this component requires $\left\langle B^{-1} \int_{-1}^{1} P_2(\xi) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, \overline{f}_{\mathrm{f}}\right) \mathrm{d}\xi \right\rangle$. An important advantage of this $\left\langle B^{-1} \int_{-1}^{1} P_2(\xi) \mathrm{d}\xi \right\rangle$ integral of the DKE is that the third Legendre order component $\int_{-1}^{1} P_3(\xi) f_a \mathrm{d}\xi$ generated by various mechanisms is excluded. The second reason is that the required parallel/perpendicular flow moments of the fast ions for the P-S and the classical diffusions and the current are determined by $\left\langle \int v^k P_2(\xi) f_{\mathrm{f}} \mathrm{d}^3\mathbf{v} \right\rangle$ and $\left\langle \int v^k P_2(\xi) f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}/B^2 \right\rangle$ with $k = -1, 1, 2, 4, 6$ as shown in Sec.IV. Since these $\int v^k P_2(\xi) f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}$ integrals in the tangential NBI operations are moderately varying functions in the (θ,ζ) space ($\int v^k P_2(\xi) f_{\rm f} {\rm d}^3\mathbf{v} \propto B^{\pm 1}$ at most) as discussed at the end of this section, both types of the surface-averaging are obtained by approximations $\left\langle \int v^k P_2(\xi) f_{\rm f} {\rm d}^3\mathbf{v} \right\rangle \cong \left\langle \int v^k P_2(\xi) f_{\rm f} {\rm d}^3\mathbf{v}/B \right\rangle / \langle B^{-1} \rangle$ and $\left\langle \int v^k P_2(\xi) f_{\rm f} {\rm d}^3\mathbf{v}/B^2 \right\rangle \cong \left\langle \int v^k P_2(\xi) f_{\rm f} {\rm d}^3\mathbf{v}/B \right\rangle \langle B^{-1} \rangle$. Therefore, we should find the solution of

$$\left(V_\parallel + C_{\rm f}^{\rm A}\right) f_{\rm A} = \frac{H_2(v) P_2(\xi)}{\tau_{\rm S} B(\theta,\zeta)}. \tag{7}$$

Here, $H_2(v)$ can be arbitrary functions of energy having finite values of $\left[v^2 H_2(v)\right]_{v=0}$ such as $x_a G(x_a)$, $\frac{4}{\sqrt{\pi}} x_a^{-3} \int_0^{x_a} y^4 \exp(-y^2)\,{\rm d}y$ discussed in Sec.III, and v^k with $k = -1, 1, 2, 4, 6$ in Sec.IV. The previous application of this method by Taguchi was a calculation of the fast ions' parallel particle flux $\left\langle B n_{\rm f} u_{\parallel \rm f} \right\rangle$ by solving $\left(V_\parallel + C_{\rm f}^{\rm A}\right) f_{\rm A} = B v \xi / \tau_{\rm S}$.[20] In this past application, the solution $f_{\rm A}$ being an odd function of $v_\parallel$ existed only in $0 \le \lambda \le 1$, and the result agrees with the $\left\langle B \int F(v) \xi f_{\rm f} {\rm d}^3\mathbf{v} \right\rangle$ integral formula in Ref.9. In contrast to this previous calculation, the solution of Eq.(7) for investigating the anisotropy exists in the full pitch-angle range. However, we need only the solution in $0 \le \lambda < 1$ as long as our purpose is in the tangential NBI operations. As a preparation for the solving procedure of Eq.(7), we shall define a function $\mathcal{V}(v)$ for each flux-surface by

$$\ln\mathcal{V}(v) \equiv 3 v_{\rm c}^3 \int \frac{{\rm d}v}{v \left\{ v^2 v_{\rm Te} (3\sqrt{\pi}/2) G(x_{\rm e}) + v_{\rm c}^3 \right\}}. \tag{8}$$

Although various energy (v) space integrals including the function $\left\{ v^2 v_{\rm Te} (3\sqrt{\pi}/2) G(x_{\rm e}) + v_{\rm c}^3 \right\}^{-1}$ will appear when handling the fast ions in the NBI-heated and/or the burning plasmas, indefinite integrals $\int v^j \left\{ v^2 v_{\rm Te} (3\sqrt{\pi}/2) G(x_{\rm e}) + v_{\rm c}^3 \right\}^{-1} {\rm d}v$ with integers in the range $j \ge -1$ such as $\ln\mathcal{V}(v)$ can be easily obtained by an approximation $G(x) \cong \left\{ (3\sqrt{\pi}/2)/x + 2x^2 \right\}^{-1}$ (a fitting formula that is exact for $x^2 \ll 1$, $x^3 \gg 1$, and $x \simeq 1$) and analytical integral formulas of $\int x^j \left(x^3 + a^3\right)^{-1} {\rm d}x$ and $\int x^{j+3} \left(x^3 + a^3\right)^{-1} {\rm d}x$. By this method, we find that a basic characteristic of Eq.(8) is $\mathcal{V}(v) \simeq v^3 / \left(v^3 + v_{\rm c}^3\right)$. There are some formulas related to this function $\mathcal{V}(v)$ that are derived only by the definition in Eq.(8) as follows:

$$-C_{\rm f}^{\rm A} \left[P_2(\xi) \left\{ \mathcal{V}(v) \right\}^{-Z_2} \int_0^v \frac{v^2 H_2(v) \left\{ \mathcal{V}(v) \right\}^{Z_2}}{v^2 v_{\rm Te} (3\sqrt{\pi}/2) G(x_{\rm e}) + v_{\rm c}^3} {\rm d}v \right] = \frac{H_2(v) P_2(\xi)}{\tau_{\rm S}}, \tag{9}$$

$$\begin{aligned} & 3\alpha v_{\rm c}^3 F(v) + \left\{ v^2 v_{\rm Te} \frac{3\sqrt{\pi}}{2} G(x_{\rm e}) + v_{\rm c}^3 \right\} v \frac{\partial F(v)}{\partial v} \\ & = \left\{ v^2 v_{\rm Te} \frac{3\sqrt{\pi}}{2} G(x_{\rm e}) + v_{\rm c}^3 \right\} \left\{ \mathcal{V}(v) \right\}^{-\alpha} v \frac{\partial}{\partial v} \left[\left\{ \mathcal{V}(v) \right\}^{\alpha} F(v) \right], \end{aligned} \tag{10}$$

$$\{\mathcal{V}(v)\}^{-\alpha}\int_0^v \frac{v^j\{\mathcal{V}(v)\}^{\alpha}\mathrm{d}v}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e})+v_\mathrm{c}^3} \cong \frac{j+1}{j+1+3\alpha}\int_0^v \frac{v^j \mathrm{d}v}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e})+v_\mathrm{c}^3}$$
$$\text{for } j\geq 0,\ \alpha\geq 0, \text{ and } v^3 \ll v_\mathrm{c}^3 \text{ where } \int_0^v \frac{v^j \mathrm{d}v}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e})+v_\mathrm{c}^3} \cong \frac{1}{j+1}\frac{v^{j+1}}{v_\mathrm{c}^3}, \tag{11}$$

$$3v_\mathrm{c}^3\int_0^v \frac{\{\mathcal{V}(v)\}^{\alpha}\mathrm{d}v}{v\{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e})+v_\mathrm{c}^3\}} = \frac{\{\mathcal{V}(v)\}^{\alpha}}{\alpha} \text{ for } \alpha>0. \tag{12}$$

Eq.(7), which we should solve, can be rewritten by using Eq.(9) and $P_2(\xi)=1-\frac{3}{2}\lambda B/B_\mathrm{M}$ as follows:

$$f_\mathrm{A}(\theta,\zeta,v,\lambda) = -\frac{P_2(\xi)}{B}\{\mathcal{V}(v)\}^{-Z_2}\int_0^v \frac{v^2H_2(v)\{\mathcal{V}(v)\}^{Z_2}}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e})+v_\mathrm{c}^3}\mathrm{d}v + G_\mathrm{A}(\theta,\zeta,v,\lambda),$$
$$\left(V_\parallel + C_\mathrm{f}^\mathrm{A}\right)G_\mathrm{A} = v_\parallel\left(\mathbf{b}\cdot\nabla\frac{1}{B}\right)\{\mathcal{V}(v)\}^{-Z_2}\int_0^v \frac{v^2H_2(v)\{\mathcal{V}(v)\}^{Z_2}}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e})+v_\mathrm{c}^3}\mathrm{d}v. \tag{13}$$

For this separated component $G_\mathrm{A}(\theta,\zeta,v,\lambda)$, we shall use the usual asymptotic expansion method for the long mean free path conditions $Z_2/(v_\mathrm{c}\tau_\mathrm{S}) \ll \left|(\delta B/B)^{3/2}\mathbf{b}\cdot\nabla\ln B\right|$. The 0th order of $(v\tau_\mathrm{S})^{-1}$ in the solution will have a form of

$$G_\mathrm{A}^0(\theta,\zeta,v,\lambda) = \left(\frac{1}{B}-\frac{1}{B_\mathrm{M}}\right)\{\mathcal{V}(v)\}^{-Z_2}\int_0^v \frac{v^2H_2(v)\{\mathcal{V}(v)\}^{Z_2}}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e})+v_\mathrm{c}^3}\mathrm{d}v + g_\mathrm{A}(v,\lambda).$$

The integration constant condition for the first term $\propto 1/B-1/B_\mathrm{M}$ is chosen to minimize both this first term and the second term $g_\mathrm{A}(v,\lambda)$ as the integration constant simultaneously. Then the solubility condition $\left\langle\left(B/v_\parallel\right)C_\mathrm{f}^\mathrm{A}G_\mathrm{A}^0\right\rangle=0$ for Eq.(13) in the circulating pitch-angle $0\leq\lambda\leq 1$ is

$$-\left\{v^2 v_{\mathrm{Te}}\frac{3\sqrt{\pi}}{2}G(x_\mathrm{e})+v_\mathrm{c}^3\right\}v\frac{\partial g_\mathrm{A}(v,\lambda)}{\partial v} + 2Z_2v_\mathrm{c}^3\left\langle\frac{B}{B_\mathrm{M}}\frac{v}{v_\parallel}\right\rangle^{-1}\frac{\partial}{\partial\lambda}\lambda\left\langle\frac{v_\parallel}{v}\right\rangle\frac{\partial g_\mathrm{A}(v,\lambda)}{\partial\lambda}$$
$$= \frac{1}{B_\mathrm{M}}\left\langle\frac{B}{B_\mathrm{M}}\frac{v}{v_\parallel}\right\rangle^{-1}\left\langle\frac{v}{v_\parallel}\left(1-\frac{B}{B_\mathrm{M}}\right)\right\rangle\left\{v^2 v_{\mathrm{Te}}\frac{3\sqrt{\pi}}{2}G(x_\mathrm{e})+v_\mathrm{c}^3\right\}$$
$$\times v\frac{\partial}{\partial v}\left\{\{\mathcal{V}(v)\}^{-Z_2}\int_0^v \frac{v^2H_2(v)\{\mathcal{V}(v)\}^{Z_2}}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e})+v_\mathrm{c}^3}\mathrm{d}v\right\}.$$

Although this function $g_\mathrm{A}(v,\lambda)$ also exists in the full pitch-angle range $0\leq\lambda\leq B_\mathrm{M}/B$ and its trapped range $\lambda>1$ is determined by bounce-integrals instead of this surface-averaging, we need only to know $\lambda<1$ of this function. We shall use the eigenfunctions $\Lambda_n(\lambda)$ with the eigenvalues κ_n in Ref.9 for this purpose. They are defined by

$$2\left\langle\frac{B}{B_\mathrm{M}}\frac{v}{v_\parallel}\right\rangle^{-1}\frac{\partial}{\partial\lambda}\lambda\left\langle\frac{v_\parallel}{v}\right\rangle\frac{\partial\Lambda_n}{\partial\lambda} = -\kappa_n\Lambda_n$$
$$\text{in } 0\leq\lambda\leq 1,\ \Lambda_n(0)=1,\ \Lambda_n(1)=0. \tag{14}$$

Since $\left\langle \left(v/v_{\|}\right)\left(1-B/B_{\mathrm{M}}\right)\right\rangle = \sigma \left\langle \left(1-B/B_{\mathrm{M}}\right)^{1/2}\right\rangle$ at $\lambda = 1$ is finite, and $\left\langle B/v_{\|}\right\rangle$ is often singular at $\lambda = 1$ (analogous to the logarithmic singularity of the complete elliptic integral of 1st kind), their ratio can be expressed by the appropriately truncated orthogonal expansion[9] using the eigenfunction $\Lambda_n\left(\lambda\right)$ as follows:

$$\left\langle \frac{B}{B_{\mathrm{M}}}\frac{v}{v_{\|}}\right\rangle^{-1}\left\langle \frac{v}{v_{\|}}\left(1-\frac{B}{B_{\mathrm{M}}}\right)\right\rangle$$
$$=\sum_n \Lambda_n(\lambda)\left\langle \left(\frac{B_{\mathrm{M}}}{B}-1\right)\int_0^1 \Lambda_n \frac{\partial\left(1-\lambda B/B_{\mathrm{M}}\right)^{1/2}}{\partial\lambda}\mathrm{d}\lambda\right\rangle \Big/ \left\langle \int_0^1 \Lambda_n^2 \frac{\partial\left(1-\lambda B/B_{\mathrm{M}}\right)^{1/2}}{\partial\lambda}\mathrm{d}\lambda\right\rangle. \tag{15}$$

By using this expansion, Eq.(13) becomes an ordinary differential equation for $g_{\mathrm{A}n}\left(v\right)$ in a series expression $g_{\mathrm{A}}\left(v,\lambda\right)=\sum\limits_n g_{\mathrm{A}n}\left(v\right)\Lambda_n(\lambda)$. Using Eq.(10), and an integration by part for $\int \mathrm{d}v$ with Eq.(12), we find that the 0th order of $\left(v\tau_{\mathrm{S}}\right)^{-1}$ in the solution of Eq.(7) at $\lambda<1$ is

$$f_{\mathrm{A}}\left(s,v=v_{\mathrm{b}},\lambda<1\right)$$
$$=-\frac{1}{B_{\mathrm{M}}}\left(1-\frac{3}{2}\lambda\right)\int_0^{v_{\mathrm{b}}}\frac{v^2 H_2(v)}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3}\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_{\mathrm{b}})}\right\}^{Z_2}\mathrm{d}v$$
$$-\frac{1}{B_{\mathrm{M}}}\int_0^{v_{\mathrm{b}}}\frac{v^2 H_2(v)}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3}\sum_n \frac{\left\langle \left(B_{\mathrm{M}}/B-1\right)\int_0^1 \Lambda_n\left\{\partial\left(1-\lambda B/B_{\mathrm{M}}\right)^{1/2}/\partial\lambda\right\}\mathrm{d}\lambda\right\rangle}{\left\langle \int_0^1 \Lambda_n^2\left\{\partial\left(1-\lambda B/B_{\mathrm{M}}\right)^{1/2}/\partial\lambda\right\}\mathrm{d}\lambda\right\rangle}\Lambda_n\left(\lambda\right)$$
$$\times\left[\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_{\mathrm{b}})}\right\}^{Z_2}+\frac{\kappa_n}{\kappa_n-3}\left\{\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_{\mathrm{b}})}\right\}^{Z_2\kappa_n/3}-\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_{\mathrm{b}})}\right\}^{Z_2}\right\}\right]\mathrm{d}v. \tag{16}$$

In the integrations by parts, Eq.(11) is used for the boundary condition at $v=0$. Since the fast ions source term is a delta function $\propto \delta\left(v-v_{\mathrm{b}}\right)$ in the energy space, only this $\int_0^{v_{\mathrm{b}}}\mathrm{d}v$ definite integral is required for Eq.(5). A function $x+\left(x^{\kappa_n/3}-x\right)\kappa_n/\left(\kappa_n-3\right)$ for $0\le x\le 1$ and $\kappa_n\ge 1$ that is included in the part expressed by a series $\sum\limits_n$ of the eigenfunctions is shown in Fig.1. In this calculation, a relation $\left[\left(x^{\alpha+1}-x\right)/\alpha\right]_{\alpha\to 0}=x\ln x$ also is used.

In the eigenfunction method, the parallel guiding center motions and the PAS effects in toroidal configurations with finite modulations $\mathbf{B}\cdot\nabla B\neq 0$, which can be measured by the reduction of a pitch-angle integral $\int_0^1 \lambda\left\langle\left(1-\lambda B/B_{\mathrm{M}}\right)^{1/2}\right\rangle^{-1}\mathrm{d}\lambda$, are expressed as an increase of the eigenvalues κ_n from $n(2n-1)$ of the usual Legendre polynomials $P_{2n-1}(\xi)$. In the summation $\sum\limits_n$ in Eq.(16) including this $x+\left(x^{\kappa_n/3}-x\right)\kappa_n/\left(\kappa_n-3\right)$, terms with large eigenvalues $\kappa_n\gg 1$ being localized at $v\sim v_{\mathrm{b}}$ cannot effectively contribute to the $\left\langle \int v^k P_2(\xi) f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\right\rangle$ integrals with $k\sim 1$ that will be calculated in Secs.III-IV by substi-

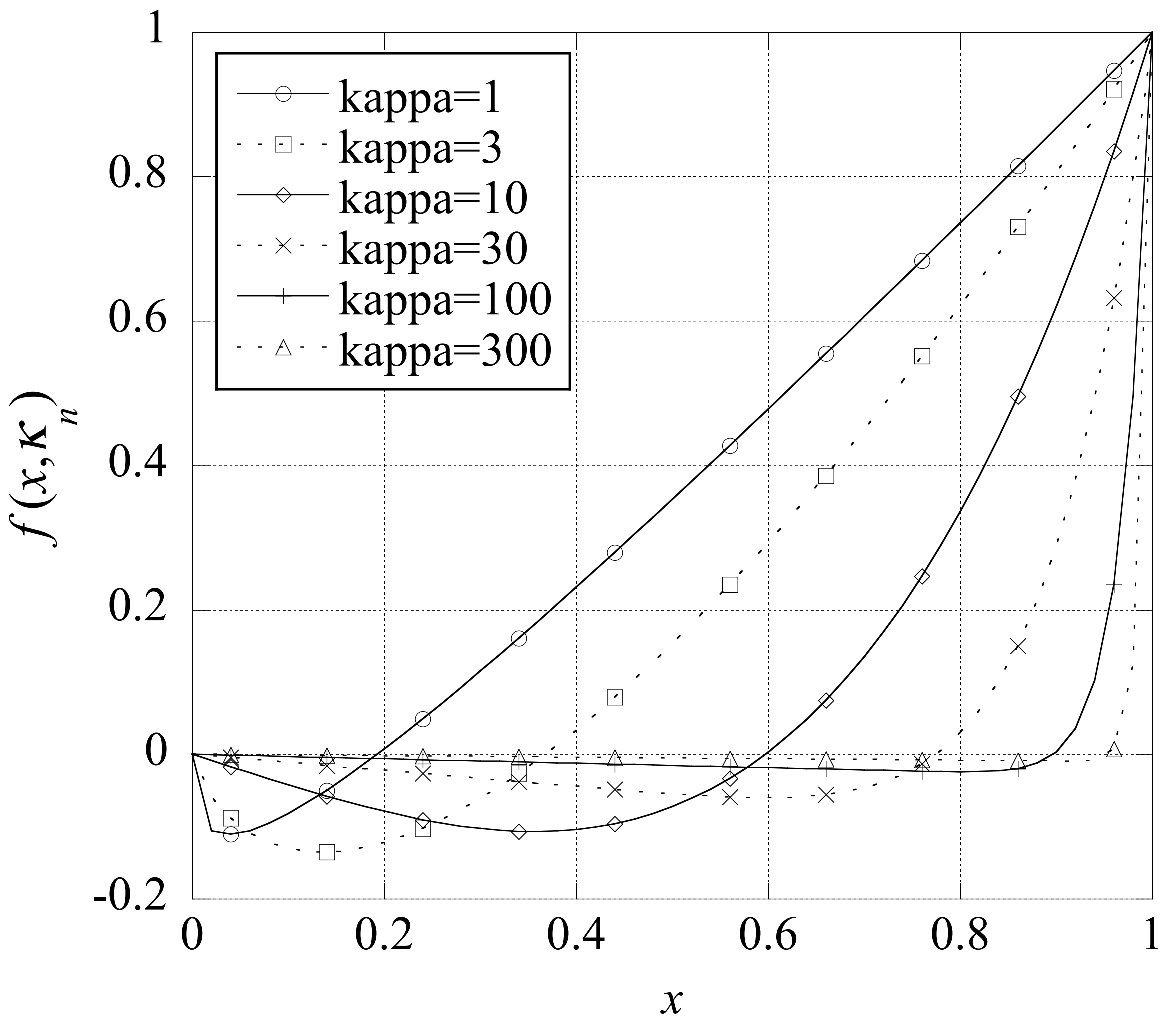

FIG. 1. The function $f(x, \kappa_n) = x + \left(x^{\kappa_n/3} - x\right)\kappa_n / \left(\kappa_n - 3\right)$.

tuting Eq.(16) into Eq.(5) with $\sigma_{\mathrm{A}} = H_2(v)P_2(\xi)/B/\tau_{\mathrm{S}}$. This is one reason for truncating the series $\sum_n$ appropriately. Analogous to the previous investigation of the surface-averaged parallel friction integrals[9,19] $\left\langle B \int v\xi L_j^{(3/2)}(x_a^2) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right) \mathrm{d}^3\mathbf{v} \right\rangle$, the six eigenvalue numbers $1 \leq n \leq 6$ are used also in this study.

Since this adjoint equation method gives only the quantities with the velocity space integral and the surface-averaging $\left\langle \int \mathrm{d}^3\mathbf{v} \right\rangle$, a change of the velocity distribution $\overline{f}_{\mathrm{f}}^{(\mathrm{even})}(\mathbf{x}, v, \lambda)$ caused by the existence of the operator $V_\parallel$ with $\mathbf{b}\cdot\nabla \ln B \neq 0$ in Eq.(1) cannot be investigated directly. To investigate this contribution of $V_\|$, we shall consider a method for comparing a calculation removing $V_\|$ and that includes this DKE term by using Eqs.(5) and (16). We shall assume fast ion sources localized at $\lambda \ll 1$ since accurate calculations of the anisotropy are required in those cases. There will not be this requirement if the beam ionization pitch-angle is $\lambda \approx 2/3$. In addition to this assumption, we previously clarified that $S_{\mathbf{x}\lambda}(\mathbf{x}, \sigma, \lambda)$ in the source term $S_{\mathbf{x}\lambda}(\mathbf{x}, \sigma, \lambda)\delta(v - v_\mathrm{b})/v^2$ must be a function of only (s, σ, λ) in $0 \leq \lambda \leq 1$ because of a consistency of the $\mathbf{B}$- and the $\mathbf{J}$-vector fields in the MHD equilibrium, and of a characteristic of the initial drift orbit conserving the magnetic moment just after the beam ionization.[9] Therefore, analogous to this previous momentum input calculation, we shall use a delta function approximation of the source term

$$\begin{aligned}
S_{\mathbf{x}\lambda}(s, \sigma = 1, \lambda)\delta(v - v_\mathrm{b})/v^2 &= 2S_0 \frac{B_\mathrm{M}\delta(\lambda - \lambda_\mathrm{b})}{\langle B(1 - \lambda_\mathrm{b}B/B_\mathrm{M})^{-1/2}\rangle} \frac{\delta(v - v_\mathrm{b})}{v^2} \\
&= S_0 \frac{B(1 - \lambda_\mathrm{b}B/B_\mathrm{M})^{-1/2}}{\langle B(1 - \lambda_\mathrm{b}B/B_\mathrm{M})^{-1/2}\rangle} \delta\left[\xi - (1 - \lambda_\mathrm{b}B/B_\mathrm{M})^{1/2}\right] \frac{\delta(v - v_\mathrm{b})}{v^2}, \\
S_{\mathbf{x}\lambda}(s, \sigma = -1, \lambda)\delta(v - v_\mathrm{b}) = 0,\ &S_0 \equiv \left\langle \int_0^1 S_{\mathbf{x}\lambda}(s, \sigma = 1, \lambda)\mathrm{d}\xi \right\rangle.
\end{aligned} \tag{17}$$

that can be obtained by appropriate Monte Carlo codes such as that in Ref.26 for tracing the initial drift motion in a time scale $2\pi R/v_\mathrm{b} \ll t \ll \tau_\mathrm{S}$ just after the beam ionization. A fixed value $\lambda_\mathrm{b} = 0.17$ is used in numerical examples in Secs.III-IV. When using this model source term, the $B_\mathrm{M}\left\langle B^{-1}\int_{-1}^{1} P_2(\xi)\mathrm{d}\xi\right\rangle$ integral of Eq.(1) is

$$\begin{aligned}
v\left\langle \frac{B_\mathrm{M}}{B}\mathbf{B}\cdot\nabla\frac{\int_{-1}^{1}\xi\overline{f}_\mathrm{f}\mathrm{d}\xi}{B}\right\rangle &= \left\langle \frac{B_\mathrm{M}}{B}\int_{-1}^{1} P_2(\xi)\left(C_\mathrm{f}\overline{f}_\mathrm{f}\right)\mathrm{d}\xi\right\rangle \\
&+ S_0 \frac{B_\mathrm{M}\left\langle (1 - \lambda_\mathrm{b}B/B_\mathrm{M})^{-1/2}\left(1 - \frac{3}{2}\lambda_\mathrm{b}B/B_\mathrm{M}\right)\right\rangle}{\langle B(1 - \lambda_\mathrm{b}B/B_\mathrm{M})^{-1/2}\rangle} \frac{\delta\left(v - v_\mathrm{b}\right)}{v^2}.
\end{aligned} \tag{18}$$

Here, a formula

$$\begin{aligned}
&V_\| \left\{P_l(\xi)F(\mathbf{x}, v)\right\} \\
&= v\frac{l}{2l+1}P_{l-1}(\xi)B^{(l+1)/2}\mathbf{b}\cdot\nabla\frac{F}{B^{(l+1)/2}} + v\frac{l+1}{2l+1}P_{l+1}(\xi)\frac{1}{B^{l/2}}\mathbf{b}\cdot\nabla\left(FB^{l/2}\right)
\end{aligned} \tag{19}$$

for Legendre polynomials $P_l(\xi)$ is convenient not only for this kind of pitch-angle integral of the DKEs for fast ions, but also for various derivation steps of various formulas for thermalized particles in Sec.III. An integration by part $\langle H\mathbf{B}\cdot\nabla F\rangle = -\langle F\mathbf{B}\cdot\nabla H\rangle$ for arbitrary scalar quantities $F(\mathbf{x})$ and $H(\mathbf{x})$ also is used to derive this LHS. In the lowest order of $(v\tau_{\mathrm{S}})^{-1}$ in the asymptotic expansion method for the long mean free path conditions, the 1st Legendre order must be $\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}^{0}\mathrm{d}\xi \propto B(\theta,\zeta)$ (i.e., a function with a symmetric phase $F(-\theta,-\zeta)=F(\theta,\zeta)$ in toroidal configurations with the stellarator symmetry $B(-\theta,-\zeta)=B(\theta,\zeta)$), and thus vanishes in the LHS of Eq.(18). In the next order of $(v\tau_{\mathrm{S}})^{-1}$, there will be a component of the odd function $\overline{f}_{\mathrm{f}}^{(\mathrm{odd})}(\mathbf{x},v,\sigma,\lambda)$ with the anti-symmetric phase $F(-\theta,-\zeta)=-F(\theta,\zeta)$ in the real space that is caused by the poloidal/toroidal variations of the slowing down collision for $\overline{f}_{\mathrm{f}}^{(l=0)}$. The generation of this component in $\left\langle B^{-1}\mathbf{B}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi/B\right)\right\rangle = \left\langle B^{-1}\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)\mathbf{b}\cdot\nabla\ln B\right\rangle$ is analogous to the "parallel viscosity force "$\left\langle\left(\int_{-1}^{1}P_2(\xi)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)\mathbf{B}\cdot\nabla\ln B\right\rangle$ of the fast ions themselves in the previous momentum input calculation. To measure this contribution of $V_{\|}$ that is included in results of Eq.(5), we compare the results with calculations omitting the LHS of Eq.(18). Analogously to the usual Legendre polynomial expansion method[28] for $\mathbf{b}\cdot\nabla\ln B=0$, the solution omitting the LHS is given by

$$\begin{aligned}&\left\langle\frac{B_{\mathrm{M}}}{B}\int_{-1}^{1}P_2(\xi)\overline{f}_{\mathrm{f}}^{(\mathbf{b}\cdot\nabla B=0)}\mathrm{d}\xi\right\rangle\\ &=S_0\tau_{\mathrm{S}}\frac{B_{\mathrm{M}}\left\langle(1-\lambda_{\mathrm{b}}B/B_{\mathrm{M}})^{-1/2}\left(1-\frac{3}{2}\lambda_{\mathrm{b}}B/B_{\mathrm{M}}\right)\right\rangle}{\langle B(1-\lambda_{\mathrm{b}}B/B_{\mathrm{M}})^{-1/2}\rangle}\frac{H(v_{\mathrm{b}}-v)}{v^2v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_{\mathrm{e}})+v_{\mathrm{c}}^3}\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_{\mathrm{b}})}\right\}^{Z_2}\\ &\qquad\text{for }\left\langle B^{-1}\mathbf{B}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi/B\right)\right\rangle=0\end{aligned}\tag{20}$$

with the unit step function $H(v_{\mathrm{b}}-v)$. This is an artificial function only for the comparison with Eq.(5) that will be shown for the $\left\langle\int x_a^2P_2(\xi)L_j^{(5/2)}(x_a^2)C_{a\mathrm{f}}(f_{a\mathrm{M}},f_{\mathrm{f}})\,\mathrm{d}^3\mathbf{v}/B\right\rangle$ integrals with $a\neq$ e,f in Sec.III and the $\langle\int v^kP_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\rangle$ integrals in Sec.IV. This comparison will clarify that the parallel guiding center motion described by the $V_{\|}$ operator reduces not only the previously investigated 1st Legendre order moments $\langle B\int F(v)\xi f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\rangle$ but also these 2nd Legendre order moments $\langle\int H_2(v)P_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\rangle$ depending on the **B**-field strength modulation on the flux-surfaces. However, these reductions are quantitatively different since the even function $\overline{f}_{\mathrm{f}}^{(\mathrm{even})}(\mathbf{x},v,\lambda)$ can be broadened to the full pitch-angle range $0\leq\lambda\leq B_{\mathrm{M}}/B$ while the odd function $\overline{f}_{\mathrm{f}}^{(\mathrm{odd})}(\mathbf{x},v,\sigma,\lambda)$ is limited in the circulating range $0\leq\lambda\leq 1$. The reduction of the anisotropy is order of $1-\langle B\rangle/B_{\mathrm{M}}$ at most.

Although the adjoint equation method does not give us the phase space structure of $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$ itself, one clear fact is that the lowest order of $(v\tau_{\mathrm{S}})^{-1}$ must be $V_{\|}\overline{f}_{\mathrm{f}}=0$ in Eq.(1).

The previous investigation of the first Legendre order moments $\left\langle B\int F(v)\xi f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\right\rangle$ also was based on this fact.[9,21] Because of this constraint, the basic structure of $\overline{f}_{\mathrm{f}}^{(\mathrm{even})}(\mathbf{x},v,\lambda)$ in the (θ,ζ,λ) space at a low energy range $v<v_{\mathrm{c}}$ where the higher Legendre order structures $\overline{f}_{\mathrm{f}}^{(l)}\equiv\frac{2l+1}{2}P_l(\xi)\int_{-1}^{1}P_l(\xi)\overline{f}_{\mathrm{f}}\mathrm{d}\xi$ with $l\geq 4$ are suppressed will be

$$\overline{f}_{\mathrm{f}}^{(\mathrm{even})}(\theta,\zeta,v,\lambda)-\left\langle f_{\mathrm{f}}^{(l=0)}\right\rangle\propto\frac{P_2(\xi)}{B}-\frac{1}{B}+\left\langle\frac{1}{B}\right\rangle=\left\langle\frac{1}{B}\right\rangle-\frac{3}{2}\frac{\lambda}{B_{\mathrm{M}}}$$

on each flux-surface that satisfies $V_{\|}\overline{f}_{\mathrm{f}}^{(\mathrm{even})}=0$. The collision term $C_{a\mathrm{f}}\left(f_{a\mathrm{M}},\overline{f}_{\mathrm{f}}^{(l=2)}\right)$ in the DKEs for thermalized ions $a\neq$ e,f that will be investigated in Sec.III and $\int v^{-1}P_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}$ of fast ions themselves in Sec.IV are integrals operations, in which only this low energy range of $\overline{f}_{\mathrm{f}}^{(\mathrm{even})}(\theta,\zeta,v,\lambda)$ can contribute, and thus will have the real space structures of $\propto 1/B(\theta,\zeta)$ on the each surface. On the other hand, $C_{\mathrm{ef}}\left(f_{\mathrm{eM}},\overline{f}_{\mathrm{f}}^{(l=2)}\right)$ in the electron DKE and $\int v^kP_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}$ with $k\geq 1$ are different integral operations that are determined by the full energy range $0\leq v\leq v_{\mathrm{b}}$ of the $\overline{f}_{\mathrm{f}}^{(\mathrm{even})}(\theta,\zeta,v,\lambda)$. In the high-energy region $v_{\mathrm{c}}<v\leq v_{\mathrm{b}}$, the existence of this velocity distribution function will be still limited to a pitch-angle range of $\lambda\ll 1$. (Recall that our interest is not in beam ionization pitch-angles of $\lambda_{\mathrm{b}}\sim 2/3$.) For this type of function depending only on (s,v,λ) in $\lambda<1$, the 2nd Legendre order becomes

$$\begin{aligned}\int_{-1}^{1}F(\lambda)\,P_2(\xi)\mathrm{d}\xi&=\frac{B}{2B_{\mathrm{M}}}\sum_{\sigma}\int_0^1F(\lambda)\left(1-\frac{3}{2}\lambda\frac{B}{B_{\mathrm{M}}}\right)\left(1-\lambda\frac{B}{B_{\mathrm{M}}}\right)^{-1/2}\mathrm{d}\lambda\\&\simeq\frac{B}{2B_{\mathrm{M}}}\sum_{\sigma}\int_0^1F(\lambda)\left(1-\lambda\frac{B}{B_{\mathrm{M}}}\right)\mathrm{d}\lambda\text{ for }\lambda\ll 1.\end{aligned}$$

Therefore, the real space structures of the integrals will deviate from the form $\propto 1/B(\theta,\zeta)$ and instead will approach different forms $C_{\mathrm{ef}}\left(f_{\mathrm{eM}},\overline{f}_{\mathrm{f}}^{(l=2)}\right)\propto P_2(\xi)B(\theta,\zeta)$ and $\int v^kP_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\propto B(\theta,\zeta)$ when the high-energy range $v_{\mathrm{c}}<v\leq v_{\mathrm{b}}$ contributes to them. These characteristics of the real space structures must be taken into account in solving procedures for the DKEs for thermalized target plasmas (Sec.III), and in the estimation of $\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi$ in Eq.(18) (Sec.IV).

III. ANISOTROPIC HEATING ANALYSIS

A. Collision between thermalized particle species

In this section, we shall consider the difference between the previously investigated part and the newly added anisotropic heating part in the DKE

$$
\begin{aligned}
&\left(V_{\|}+V_E\right) f_{a1}-\sum_{b\neq\mathrm{f}} C_{ab}\left(\overline{f}_a,\overline{f}_b\right)\\
&=\left(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s\right)\left\{X_{a1}-X_{a2}L_1^{(3/2)}(x_a^2)\right\}\frac{f_{a\mathrm{M}}}{\langle T_a\rangle}+C_{a\mathrm{f}}\left(f_{a\mathrm{M}},\overline{f}_{\mathrm{f}}\right)+v\xi e_a\frac{B\left\langle\mathbf{B}\cdot\mathbf{E}^{(\mathbf{A})}\right\rangle}{\langle B^2\rangle}\frac{f_{a\mathrm{M}}}{\langle T_a\rangle}, \qquad (21)\\
&\qquad X_{a1}=-\frac{1}{\langle n_a\rangle}\frac{\partial\langle p_a\rangle}{\partial s}-e_a\frac{\partial\Phi}{\partial s},\ X_{a2}=-\frac{\partial\langle T_a\rangle}{\partial s}
\end{aligned}
$$

for the thermalized particles' gyro-phase velocity distribution $\overline{f}_a=f_{a\mathrm{M}}+f_{a1}$. Here, analogously to Sec.II, $\mathbf{v}_{\mathrm{d}a}=(c/e_a)\left(m_a v_{\|}^2/B+\mu\right)\mathbf{b}\times\nabla\mathrm{ln}B$ is the perpendicular guiding center velocity, and $\frac{\partial}{\partial s}\langle p_a\rangle$, $\frac{\partial}{\partial s}\langle T_a\rangle$ in X_{a1}, X_{a2} correspond to $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial f_{a\mathrm{M}}/\partial s$ given by the differential keeping constant (v,ξ) that is mentioned in Sec.II. The radial electric field $\frac{\partial}{\partial s}\Phi$ in X_{a1} corresponds to $V_E f_{a\mathrm{M}}$ given by Eq.(6). This equation is solved under a constraint $\left\langle\int_{-1}^{1}f_{a1}\mathrm{d}\xi\right\rangle=0$ for the full (θ,ζ,ξ) range as the definition of the density $\langle n_a\rangle$ (number of particles) and the pressure $\langle p_a\rangle$ (energy) of the species a. We must take care of the fact that we already calculated responses to these radial gradient forces $\frac{\partial}{\partial s}\langle p_a\rangle$, $m_a\frac{\partial}{\partial s}\langle r_a\rangle=\frac{5}{2}\frac{\partial}{\partial s}\langle p_a T_a\rangle$, $\frac{\partial}{\partial s}\Phi$, and the parallel force terms $E_{\|}^{(\mathbf{A})}$, $C_{a\mathrm{f}}\left(f_{a\mathrm{M}},f_{\mathrm{f}}^{(l=1)}\right)$ in this RHS before adding the anisotropic heating source term $C_{a\mathrm{f}}\left(f_{a\mathrm{M}},f_{\mathrm{f}}^{(l=2)}\right)$. Here, $f_a^{(l)}\equiv P_l(\xi)\frac{2l+1}{2}\int_{-1}^{1}P_l(\xi)\overline{f}_a\mathrm{d}\xi$ is the Legendre expansion term of order l in the gyro-phase-averaged velocity distribution. Reasons for which we omit $C_{a\mathrm{f}}\left(f_{a1},f_{\mathrm{f}}\right)$ of collision with the fast ions is stated in Refs.9 and 19. The reason for $\int_{-1}^{1}P_l(\xi)C_{a\mathrm{f}}\left(\overline{f}_a,\overline{f}_{\mathrm{f}}\right)\mathrm{d}\xi$ with $l=0,1$ (i.e., energy/momentum input) is the conservation of energy and momentum in the combined use with Eq.(2). For higher Legendre orders $l\geq 2$, the relation $\left|C_{a\mathrm{f}}\left(f_{a1},f_{\mathrm{f}}\right)\right|\ll\left|\sum_{b\neq\mathrm{f}}C_{ab}\left(f_{a1},f_{b\mathrm{M}}\right)+C_{aa}\left(f_{a\mathrm{M}},f_{a1}\right)\right|$ due to a low density of fast ions is a main reason. Therefore it also should be noted that $\left|\int_{-1}^{1}P_l(\xi)C_{a\mathrm{f}}\left(f_{a1},\overline{f}_{\mathrm{f}}\right)\mathrm{d}\xi\right|\ll\left|\int_{-1}^{1}P_l(\xi)C_{a\mathrm{f}}\left(f_{a\mathrm{M}},\overline{f}_{\mathrm{f}}\right)\mathrm{d}\xi\right|$ is not guaranteed for general Legendre orders l, and thus to retain $C_{a\mathrm{f}}\left(f_{a\mathrm{M}},f_{\mathrm{f}}^{(l)}\right)$ with too large l values in Eq.(21) is meaningless. This is one reason why we restrict our investigation of the fast-ion-driven effects only to $C_{a\mathrm{f}}\left(f_{a\mathrm{M}},f_{\mathrm{f}}^{(l\leq 2)}\right)$. For the friction collision term $C_{a\mathrm{f}}\left(f_{a\mathrm{M}},f_{\mathrm{f}}^{(l=1)}\right)$, we should include not only the previously investigated Sonine (generalized Laguerre) polynomial expansion coefficients of $\left\langle B\int_{-1}^{1}\xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}},\overline{f}_{\mathrm{f}}\right)\mathrm{d}\xi\right\rangle$,[6] but also those of the poloidal and toroidal variations $\int_{-1}^{1}\xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}},\overline{f}_{\mathrm{f}}\right)\mathrm{d}\xi-\left\langle B\int_{-1}^{1}\xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}},\overline{f}_{\mathrm{f}}\right)\mathrm{d}\xi\right\rangle B/\left\langle B^2\right\rangle\propto\widetilde{U}\left(\theta,\zeta\right)$ for which an obtaining method will be shown in Sec.IV. Essential differences between the DKEs for fast ions (Sec.II) and for thermalized particles (this section) appear in the LHS of Eq.(21). One difference is in the linearization $C_{ab}\left(\overline{f}_a,\overline{f}_b\right)=C_{ab}\left(f_{a1},f_{b\mathrm{M}}\right)+C_{ab}\left(f_{a\mathrm{M}},f_{b1}\right)$ of the collisions between thermalized particle species $a,b\neq\mathrm{f}$, in which $C_{ab}\left(f_{a1},f_{b\mathrm{M}}\right)$ is a differential operator for the test particles' f_{a1} while $C_{ab}\left(f_{a\mathrm{M}},f_{b1}\right)$ is an integral operator for the field particles' f_{b1}, and the other is the explicit inclusion of the $\mathbf{E}\times\mathbf{B}$ operator V_E. When this $V_\parallel+V_E$ operator is included, various pitch-angle space structures (Legendre orders) in f_{a1}, f_{b1}, f_{c1}, ... with various phases in the (θ,ζ) space are generated by the source terms in the RHS of Eq.(21). This situation is complicated in Eq.(21) rather than in Eq.(1) for the fast ions.

For avoiding confusion in considering these (θ,ζ,ξ) space structures of f_{a1}, f_{b1}, f_{c1}, ... of thermalized particles simultaneously, we shall separate the problem described by Eq.(21) into three parts: (1) viscosity, (2) Pfirsch-Schlüter (P-S), and (3) anisotropic heating. This separation is based on following characteristics of the Coulomb collisions between the thermalized particle species $a,b,c,...\neq\mathrm{f}$. One is that the field particle portion $C_{ab}\left(f_{a\mathrm{M}},f_{b1}\right)$ causing the coupling between equations for thermalized species is an integral operator and an eigenoperator of spherical harmonics suppressing its higher Legendre orders. Another reason is the unimportance of the lowest Legendre order moment of the collision term that is indicated by the relation $C_{ab}\left(f_{a\mathrm{M}},f_{b\mathrm{M}}\right)=0$ for $\langle T_a\rangle=\langle T_b\rangle$, the energy conservation of like-particle collisions $\int v^2C_{aa}\left(f_a,f_a\right)\mathrm{d}^3\mathbf{v}=0$, and the particle conservation of general colliding species pairs $\int C_{ab}\left(f_a,f_b\right)\mathrm{d}^3\mathbf{v}=0$. An important difference between Eqs.(1) and (21) exists also in the self-adjoint relations

$$\int\widehat{g}_aC_{ab}\left(\widehat{f}_af_{a\mathrm{M}},f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}=\int\widehat{f}_aC_{ab}\left(\widehat{g}_af_{a\mathrm{M}},f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v},$$
$$\int\widehat{g}_aC_{ab}\left(f_{a\mathrm{M}},\widehat{f}_bf_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}=\int\widehat{f}_bC_{ba}\left(f_{b\mathrm{M}},\widehat{g}_af_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}$$

of thermal-thermal collisions. In the neoclassical transport theory for deriving the transport matrix, however, it is required that these relations must be satisfied only in their surface-average. We can consider the separation into the three parts also based on this fact. In particular, the necessity of the field particle portion $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}\right)$ depends on velocity distribution function components with various (θ, ζ, ξ) space structures, and thus is important in this separation. Here, we summarize the necessity of $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l=0,1,2)}\right)$.

Firstly, the first Legendre moment $f_{a1}^{(l=1)}$ consists of two components with different (θ, ζ) space structures. For this kind of consideration, we shall recall Eq.(19). One component is the poloidal/toroidal variation that is determined by the lowest Legendre order ($l = 0$) term (corresponding to the particle/energy balance equations Eqs.(A14-A15)) of this equation. The other component is the integration constant $\left\langle B \int_{-1}^{1} \xi f_{a1} \mathrm{d}\xi \right\rangle B / \langle B^2 \rangle$ of this balance equation that is determined by the $\left\langle B \int_{-1}^{1} \xi \mathrm{d}\xi \right\rangle$ integral of the DKE (i.e., surface-averaged parallel force balance). Because of these different determination procedures, their v-space structures also are different. Both components exist dominantly in f_{a1} in various collisionality conditions and thus the field particle portion $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l=1)}\right)$ is a main cause of the coupling of the DKEs for different species. On this Legendre order, it also should be noted that the first Legendre moment $\int_{-1}^{1} \xi \left(V_E f_{a1}\right) \mathrm{d}\xi$ of the $\mathbf{E} \times \mathbf{B}$ operator Eq.(6), and that of the ∇B and the curvature drift term $\int_{-1}^{1} \xi \left(\mathbf{v}_{\mathrm{d}a} \cdot \nabla f_{a1}\right) \mathrm{d}\xi$ must be omitted as long as we use the flux-surface coordinates system based on the MHD equilibrium where the CGL tensor formula is used with neglecting the inertia force as in Eqs.(A1-A2). Although the DKE solution is always determined under the constraint of the parallel force balance as the $\int_{-1}^{1} \xi \mathrm{d}\xi$ integral of the DKE, the $V_E + \mathbf{v}_{\mathrm{d}a} \cdot \nabla$ operators are irrelative to this force balance.

Secondly, explicit handling of the energy scattering/exchange collision for the lowest Legendre order $l = 0$ will be required only for calculating the P-S diffusions of multi-ion species plasmas.[17] In contrast to the fast ion DKE shown in Eqs.(1-2) where the poloidal/toroidal variations of the slowing down collision for ions with suprathermal energies may generate $\left\langle \mathbf{b} \cdot \nabla \left(\int_{-1}^{1} \xi \overline{f}_{\mathrm{f}} \mathrm{d}\xi / B \right) \right\rangle$, the energy exchange between unlike thermalized species approximately given by

$$
\begin{aligned}
&-m_a \int v^2 C_{ab}(f_a, f_b) \mathrm{d}^3\mathbf{v} = m_b \int v^2 C_{ba}\left(f_b, f_a\right) \mathrm{d}^3\mathbf{v} \\
&\cong 6 \frac{n_a}{\tau_{ab}} \frac{m_a}{m_b} \left\{ 1 + \left(\frac{v_{\mathrm{T}b}}{v_{\mathrm{T}a}} \right)^2 \right\}^{-3/2} \left(\frac{p_a}{n_a} - \frac{p_b}{n_b} \right) = 6 \frac{n_b}{\tau_{ba}} \frac{m_b}{m_a} \left\{ 1 + \left(\frac{v_{\mathrm{T}a}}{v_{\mathrm{T}b}} \right)^2 \right\}^{-3/2} \left(\frac{p_a}{n_a} - \frac{p_b}{n_b} \right)
\end{aligned}
$$

is only a minor function of the collision operator. Therefore the $C_{ab}\left(f_{a1}^{(l=0)}, f_{b\mathrm{M}}\right) +$

$C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l=0)}\right)$ as a cause of the poloidal/toroidal variation of $f_{a1}^{(l=1)}$ in Eq.(19) is explicitly included only in a collisional limit where the poloidal/toroidal variations of the local temperatures p_a/n_a, p_b/n_b may become dominant components of f_{a1}, f_{b1}. It also should be noted that, even for the fast ions, this $\mathbf{B}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi/B\right)$ in the suprathermal velocity range does not effectively contribute to the poloidal/toroidal variations of the friction integrals as explained in Sec.IV (negligible in comparison with the diamagnetic flux divergence described by $(\mathbf{v}_{\mathrm{df}}\cdot\nabla s)\,\partial\overline{f}_{\mathrm{f}}/\partial s$). Also for the thermalized particles' DKEs, in which our purpose is in the determination of the $f_a(\mathbf{x},\mathbf{v})$ in the thermalized velocity range $m_a v^2 \sim 2T_a$ within an accuracy where the first few Laguerre orders in each Legendre moments are correct, the parallel flow divergence due to the lowest order Legendre moment $\int_{-1}^{1}C_{ab}\left(\overline{f}_a, \overline{f}_b\right)\mathrm{d}\xi$ with the aforementioned characteristics is not important. Analogously, only for the P-S diffusion calculation, $c\nabla\Phi\times\mathbf{B}\cdot\nabla f_a^{(l=0)}$ in the V_E operator is regarded as substantial divergences of the parallel particle/energy fluxes that can contribute to the parallel friction collision. Since it corresponds to $\nabla\Phi\times\mathbf{B}\cdot\nabla\left(n_a/B^2\right)$ and $\nabla\Phi\times\mathbf{B}\cdot\nabla\left(p_a/B^2\right)$ in Eqs.(A14-A15) in Appendix A, it substantially contributes to the friction only when $\left|f_{a1}^{(l=0)}\right|/f_{a\mathrm{M}} \sim \left(B_{\mathrm{M}}-B_{\min}\right)/\left(B_{\mathrm{M}}+B_{\min}\right)$.

Finally, the necessity of the $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l=2)}\right)$ for the second Legendre order is considered. Although the poloidal/toroidal variations of the anisotropy $\int_{-1}^{1}P_2(\xi)f_{a1}\mathrm{d}\xi - \left\langle B^{-1}\int_{-1}^{1}P_2(\xi)f_{a1}\mathrm{d}\xi\right\rangle B^{-1}/\left\langle B^{-2}\right\rangle$ corresponding to the neoclassical viscosity tensor can become a dominant component in the $f_{a1}(\mathbf{x},\mathbf{v})$ in the long mean free path conditions of non-symmetric stellarator/heliotron plasmas, its amplitude is $\left|f_{a1}^{(l=2)}\right| \sim \left|f_{a1}^{(l=1)}\right|$ at most as long as the ambipolar condition $\langle\mathbf{J}\cdot\nabla s\rangle = 0$ is satisfied. Because of a relation

$$\frac{\int x_a^2 P_2(\xi) C_{ab}\left(f_{a\mathrm{M}},\ x_b^2 P_2(\xi) f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}}{\int x_a\xi C_{ab}\left(f_{a\mathrm{M}},\ x_b\xi f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}} = \frac{6}{5}\frac{v_{\mathrm{T}a}}{v_{\mathrm{T}b}}\left\{1+\left(\frac{v_{\mathrm{T}a}}{v_{\mathrm{T}b}}\right)^2\right\}^{-1} = \frac{6}{5}\frac{v_{\mathrm{T}b}}{v_{\mathrm{T}a}}\left\{1+\left(\frac{v_{\mathrm{T}b}}{v_{\mathrm{T}a}}\right)^2\right\}^{-1}$$

for the thermal-thermal collisions, the $l \geq 2$ field particle portion $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l\geq2)}\right)$ of unlike-particle collision $a \neq b$ is often neglected, and only $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l=1)}\right)$ is retained as the coupling between the DKEs for the thermalized species. The assumption of the ambipolar conditions with $\left|\frac{\partial}{\partial s}\Phi\right| \gg 2T_a\left|e_a^{-1}\frac{\partial}{\partial s}\ln B\right|$ in the present study is required not only for the drift approximation of Vlasov operator where $\mathbf{v}_{\mathrm{d}a}\cdot\nabla f_{a1}$ (in particular the drift being tangential to the flux-surface) is neglected as in Eq.(21), but also for this collision approximation where an appropriate suppression of the so-called $1/\nu$ diffusion of ions by this radial electric field is assumed. Based on this assumption, the poloidal/toroidal variations of $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l\geq 2)}\right)$ are still neglected in the determination of the responses to $\mathbf{v}_{\mathrm{d}a}\cdot\nabla s$, $E_{\parallel}^{(\mathbf{A})}$, and $C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=1)}\right)$ in the RHS in Eq.(21), while we add a surface-averaged component $C_{ab}\left(f_{a\mathrm{M}}, f_{b}^{(l=2)}\right) \propto P_2(\xi)/B(\theta,\zeta)$ (b=f, ions) caused by the anisotropic heating effect due to the fast ions. (The anisotropic heating source term $C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_{\mathrm{f}}^{(l=2)}\right)$ in the electron DKE may have a different real space structure as discussed in the end of Sec.II.) When this term is included, a velocity distribution function component having a real space structure satisfying the local parallel force balance in the (θ,ζ,v,ξ) space $V_{\parallel}\left(P_2(\xi)/B - 1/B + \langle 1/B\rangle\right) = V_{\parallel}\left(\langle 1/B\rangle - \frac{3}{2}\lambda/B_{\mathrm{M}}\right) = 0$ is generated as the response. Since this determination is irrelative to the first Legendre order in the velocity distribution and collision operators, the field particle portion $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l=2)}\right)$ for this component (both for like-particle collisions $a = b$ and unlike-particle collisions $a \neq b$) also is included. For collision between the electrons and the thermalized ions, however, we shall use the usual small mass ratio approximation for allowing $|T_{\mathrm{e}} - T_{\mathrm{i}}| \sim T_{\mathrm{e}}, T_{\mathrm{i}}$, in which $C_{\mathrm{ei}}\left(f_{\mathrm{eM}}, f_{\mathrm{i1}}^{(l=2)}\right)$ and $C_{\mathrm{ie}}\left(f_{\mathrm{iM}}, f_{\mathrm{e1}}^{(l=2)}\right)$ are not included. The necessity of the $l = 2$ like-particle field particle portion $C_{aa}\left(f_{a\mathrm{M}}, f_{a1}^{(l=2)}\right)$ depends also on the generation of higher Legendre orders $l \geq 3$. In the previous theories for the neoclassical viscosity such as Ref.3, the $C_{aa}\left(f_{a\mathrm{M}}, f_{a1}^{(l=2)}\right)$ is often neglected because of a relation $\left|C_{aa}\left(f_{a\mathrm{M}}, f_{a1}^{(l\geq 2)}\right)\right| \ll \left|\sum_{b\neq \mathrm{f}} C_{ab}\left(f_{a1}^{(l\geq 2)}, f_{b\mathrm{M}}\right)\right|$ due to the generation of the orders $l \geq 3$. In the anisotropic heating calculation for the additional velocity distribution component $f_{a1}^{(l=2)} \propto P_2(\xi)/B(\theta,\zeta)$, however, the $C_{aa}\left(f_{a\mathrm{M}}, f_{a1}^{(l=2)}\right)$ also is retained since the orders $l \geq 3$ are scarcely generated by the anisotropic heating source term. In this method where the $l = 2$ field particle portion $C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{(l=2)}\right)$ is neglected for the poloidal/toroidal variations while it is retained for the surface-averaged component $\propto P_2(\xi)/B(\theta,\zeta)$, the aforementioned self-adjoint relations are retained in their surface-average. The poloidal/toroidal variations corresponding to the neoclassical viscosity are not important for the anisotropic-pressure MHD equilibrium because of a relation $\left\langle p_{\perp a} - p_{\parallel a}\right\rangle \left\langle (p_{\perp a} - p_{\parallel a})/B^2\right\rangle < 0$. When only this type of the anisotropy exists, the radial gradients discussed in Appendix A are regarded as those of isotropic-pressure species. From the viewpoint of the drift approximation, this handling of the radial gradients corresponds to a neglect of $|\mathbf{v}_{\mathrm{d}a}\cdot\nabla f_{a1}| \ll \left|\left(V_{\parallel} + V_E\right) f_{a1}\right|$ for the solubility condition of the parallel particle/energy fluxes. On the other hand, this DKE term approximation is not appropriate for the velocity distribution component generated by the anisotropic heating effect $C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=2)}\right)$. If the generated anisotropy is non-negligible in Eqs.(A14-A15) (i.e., comparable or larger than the poloidal/toroidal variations), the absolute value of the radial gradient term $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial f_{a\mathrm{M}}/\partial s$ must be corrected for a consistency with the P-S current in the MHD equilibrium. This correction corresponds to a drift approximation retaining a part of $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial f_{a1}/\partial s$ analogous to the radial gradient of the fast ions' velocity distribution that was mentioned in Sec.II. (As mentioned below, not only the parallel velocity term $V_{\|}$ but also the $\mathbf{E}\times\mathbf{B}$ term V_E for this velocity distribution component $\propto P_2(\xi)/B(\theta,\zeta)$ is neglected.)

B. Separation of the DKE for thermalized particles

Based on these characteristics of the collision operator, the aforementioned three parts in Eq.(21) are considered. From the discussion based on Eq.(19) in Sec.II, it is obvious that $f_{a1}^{(l=3)}$ generated by various mechanisms is excluded in the $\left\langle B^{-1}\int_{-1}^{1}P_2(\xi)\mathrm{d}\xi\right\rangle$ integral of the DKE. The calculations in this section also utilize this fact. However, poloidal/toroidal variations of the first Legendre order $\int_{-1}^{1}\xi f_{a1}\mathrm{d}\xi-\left\langle B\int_{-1}^{1}\xi f_{a1}\mathrm{d}\xi\right\rangle B/\left\langle B^2\right\rangle$ with the anti-symmetric phase $F\left(-\theta,-\zeta\right)=-F\left(\theta,\zeta\right)$ may remain there. Even when solving Eq.(21) with excluding the anisotropic heating source term $C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=2)}\right)$, the existence of $\left\langle B^{-1}\int_{-1}^{1}P_2(\xi)\left\{\left(V_{\|}+V_E\right)\overline{f}_a\right\}\mathrm{d}\xi\right\rangle=\left\langle B^{-1}\int_{-1}^{1}P_2(\xi)\sum_{b\neq\mathrm{f}}C_{ab}\left(\overline{f}_a,\overline{f}_b\right)\mathrm{d}\xi\right\rangle$ is not forbidden. Although this quantity is irrelative to the definition of $\langle n_a\rangle$ and $\langle p_a\rangle$, it may change radial gradients discussed in Appendix A. On the operator $V_{\|}$, based on the fact that $\int_{-1}^{1}\xi\overline{f}_a\mathrm{d}\xi/B$ in

$$\left\langle\frac{B_{\mathrm{M}}}{B}\int_{-1}^{1}P_2(\xi)\left(V_{\|}\overline{f}_a\right)\mathrm{d}\xi\right\rangle=v\left\langle\frac{B_{\mathrm{M}}}{B}\left(\int_{-1}^{1}\xi\overline{f}_a\mathrm{d}\xi\right)\mathbf{b}\cdot\nabla\ln B\right\rangle=v\left\langle\frac{B_{\mathrm{M}}}{B}\mathbf{B}\cdot\nabla\frac{\int_{-1}^{1}\xi\overline{f}_a\mathrm{d}\xi}{B}\right\rangle$$

must satisfy the solubility condition $\left\langle \mathbf{B}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_a\mathrm{d}\xi/B\right)\right\rangle=0$, we investigate the lowest order $j=0$ in the Laguerre expansion of this quantity by using Eq.(A15) as follows:

$$\begin{aligned}&\left\langle \frac{B_\mathrm{M}}{B}\int v^2P_2(\xi)\left(V_\parallel\overline{f}_a\right)\mathrm{d}^3\mathbf{v}\right\rangle=\frac{2}{m_a}\left\langle \frac{B_\mathrm{M}}{B}\mathbf{B}\cdot\nabla\frac{Q_{\parallel a}}{B}\right\rangle\\&=-3cE_s\left\langle \frac{B_\mathrm{M}}{B}\left(\frac{\nabla s\times\mathbf{B}}{B^2}\cdot\nabla\ln B\right)\frac{p_a}{m_a}\right\rangle+\left\langle\frac{B_\mathrm{M}}{B}\left(\int v^2C_a(f_a)\mathrm{d}^3\mathbf{v}-\left\langle\int v^2C_a(f_a)\mathrm{d}^3\mathbf{v}\right\rangle\right)\right\rangle.\end{aligned} \tag{22}$$

On the other hand, the $\left\langle B^{-1}\int_{-1}^{1}P_2(\xi)\mathrm{d}\xi\right\rangle$ integral of the $\mathbf{E}\times\mathbf{B}$ operator in Eq.(6) is irrelative to this solubility condition, and thus given by

$$\left\langle \frac{B_\mathrm{M}}{B}\int v^2P_2(\xi)\left(V_E\overline{f}_a\right)\mathrm{d}^3\mathbf{v}\right\rangle=-\frac{cE_s}{3}\left\langle\frac{B_\mathrm{M}}{B}\left(\frac{\nabla s\times\mathbf{B}}{B^2}\cdot\nabla\ln B\right)\int v^2\left\{1+2P_2(\xi)\right\}\overline{f}_a\mathrm{d}^3\mathbf{v}\right\rangle, \tag{23}$$

which is obtained by straightforward integrations by parts. The lowest Laguerre order of the $\left\langle B^{-1}\int_{-1}^{1}P_2(\xi)\left\{\left(V_\parallel+V_E\right)\overline{f}_a\right\}\mathrm{d}\xi\right\rangle$ integral given by these methods is negligible in the viscosity part, and may be generated in the P-S part with finite radial electric fields $\partial\Phi/\partial s\neq0$ in following discussions.

Next, we shall discuss the approximation and solving procedure for the viscosity and the P-S parts. The viscosity part handles responses to a part of the radial drift term

$$\sigma_{Xa}\equiv\mathbf{v}_{\mathrm{d}a}\cdot\nabla s-\frac{c}{e_a}m_aV_\parallel\left(v_\parallel\widetilde{U}\right)=-\frac{c}{e_a}m_av^2P_2(\xi)\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\widetilde{U}\mathbf{b}\right)\cdot\nabla\ln B \tag{24}$$

and the surface-averaged parallel force term by a method in Refs.3, 5, 29, and 30. Here, $\widetilde{U}$ is the solution of $\mathbf{B}\cdot\nabla\left(\widetilde{U}/B\right)=(\mathbf{B}\times\nabla s)\cdot\nabla B^{-2}$, $\left\langle B\widetilde{U}\right\rangle=0$. The solution g_a of the separated equation

$$\begin{aligned}&\left(V_\parallel+V_E^\mathrm{DKES}\right)g_a-\sum_{b\neq\mathrm{f}}C_{ab}\left(g_a^{(l\neq0)},f_{b\mathrm{M}}\right)\\&=\sigma_{Xa}\left\{X_{a1}-X_{a2}L_1^{(3/2)}(x_a^2)\right\}\frac{f_{a\mathrm{M}}}{\langle T_a\rangle}+\frac{3}{2}\frac{\sigma_{Ua}}{m_av\langle B^2\rangle}\left\langle B\int_{-1}^{1}\xi f_{a1}\mathrm{d}\xi\right\rangle\\&+\frac{3}{2}\xi\frac{B}{\langle B^2\rangle}\left\langle B\int_{-1}^{1}\xi\left\{\sum_{b\neq\mathrm{f}}C_{ab}\left(\overline{f}_a,\overline{f}_b\right)+C_{a\mathrm{f}}\left(f_{a\mathrm{M}},\overline{f}_\mathrm{f}\right)\right\}\mathrm{d}\xi\right\rangle+v\xi e_a\frac{B\left\langle\mathbf{B}\cdot\mathbf{E}^{(\mathbf{A})}\right\rangle}{\langle B^2\rangle}\frac{f_{a\mathrm{M}}}{\langle T_a\rangle},\end{aligned} \tag{25}$$

$$\sigma_{Ua}\equiv-m_aV_\parallel(v_\parallel B)=-m_av^2P_2(\xi)\mathbf{b}\cdot\nabla B$$

is obtained by the following approximations. One approximation is that for the collision operator

$$\sum_{b\neq\mathrm{f}}\left[C_{ab}\left(g_a,f_{b\mathrm{M}}\right)+C_{ab}\left(f_{a\mathrm{M}},g_b\right)\right]\cong\sum_{b\neq\mathrm{f}}C_{ab}\left(g_a^{(l\neq0)},f_{b\mathrm{M}}\right)$$

in the LHS. In this approximation, the field particle portions $C_{ab}\left(f_{a\mathrm{M}}, g_b\right)$ are omitted and $g_a^{(l=0)}$ is excluded for the particle/energy conservation since the definition of this g_a includes $\left\langle B\int_{-1}^{1}\xi g_a \mathrm{d}\xi\right\rangle = 0$, and since various parallel flow divergences explained in Sec.III-A such as $\int_{-1}^{1} C_{ab}\left(\overline{f}_a, \overline{f}_b\right)\mathrm{d}\xi$ and $V_E f_{a1}^{(l=0)}$ are not important in this LHS handling the higher Legendre orders $g_a^{(l\geq 2)}$ as dominant components. Second is the use of

$$V_E^{\mathrm{DKES}} \equiv cE_s\left\langle B^2\right\rangle^{-1}\nabla s\times\mathbf{B}\cdot\nabla_{(v,\xi)=\mathrm{const}}$$

instead of Eq.(6) in the LHS for the solubility condition and the anti-symmetric property $\left\langle\int\widehat{g}_a\left(V_E^{\mathrm{DKES}}\widehat{f}_a f_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}\right\rangle = -\left\langle\int\widehat{f}_a\left(V_E^{\mathrm{DKES}}\widehat{g}_a f_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}\right\rangle$.[27] This approximation is justified by the fact that dominant components in the g_a are the poloidal/toroidal variations of the higher Legendre orders and therefore $\partial/\partial\xi$ and $\partial/\partial v$ in Eq.(6) become a higher order of the $\mathbf{B}$-field strength modulation on the surfaces. One important advantage of this approximation together with the further approximation $\sum\limits_{b\neq\mathrm{f}} C_{ab}\left(g_a^{(l\neq 0)}, f_{b\mathrm{M}}\right)\cong\nu_{\mathrm{D}}^a(v)\mathcal{L}g_a$ for handling the higher Legendre orders $l\geq 3$ mainly is a reduction of the phase-space dimension of the LHS to 3D (θ,ζ,ξ). Therefore, the solution g_a is given by a combination of solutions G_{Xa} and G_{Ua} of the equation

$$\left(V_\parallel + V_E^{\mathrm{DKES}} - \nu_{\mathrm{D}}^a\mathcal{L}\right)\begin{bmatrix} G_{Xa} \\ G_{Ua}\end{bmatrix} = \begin{bmatrix}\sigma_{Xa} \\ \sigma_{Ua}\end{bmatrix} \tag{26}$$

as shown in Ref.3. It also should be noted that the anisotropy relaxation Krook operator[17] $\sum\limits_{b\neq\mathrm{f}} C_{ab}\left(g_a^{(l=2)}, f_{b\mathrm{M}}\right)\cong -\nu_{\mathrm{T}}^a(v)g_a^{(l=2)}$ including the energy scattering effect, instead of this PAS operator, is used for a low energy region $\nu_{\mathrm{T}}^a/v > (8/5\pi)(4\pi^2/V')\left|\chi' m - \psi' n\right|/\left\langle B^2\right\rangle^{1/2}$ of each Fourier modes $\sin(m\theta - n\zeta)$, $\cos(m\theta - n\zeta)$ in the (θ,ζ,ξ) space for which the higher Legendre orders $l\geq 3$ are suppressed.[5] Here, $\chi'\equiv\mathrm{d}\chi/\mathrm{d}s$, $\psi'\equiv\mathrm{d}\psi/\mathrm{d}s$, and $V'\equiv\mathrm{d}V/\mathrm{d}s$ are radial gradients of the poloidal flux, the toroidal flux, and the volume enclosed by the flux-surface $s=$ const in the contravariant expression of the $\mathbf{B}$-vector field, respectively.[3,9] This problem results in an algebraic handling of $\left\langle B\int v\xi L_j^{(3/2)}(x_a^2)f_a\mathrm{d}^3\mathbf{v}\right\rangle$ using the Braginskii's matrix expression of the full linearized collision operator for the first Legendre order $l=1$, and these surface-averaged parallel flow moments included in the RHS of Eq.(25) also are determined. Since the poloidal/toroidal variations of the parallel friction $\int_{-1}^{1}\xi C_{ab}\left(\overline{f}_a, \overline{f}_b\right)\mathrm{d}\xi - \left\langle B\int_{-1}^{1}\xi C_{ab}\left(\overline{f}_a, \overline{f}_b\right)\mathrm{d}\xi\right\rangle B/\left\langle B^2\right\rangle$ are negligible in this part, the "frictionless"local parallel force balance determines the lowest Legendre order $g_a^{(l=0)}$ as a minor component. A basic characteristic of this component is easily understood by approximated relations

$$p_a - \langle p_a \rangle \cong -\frac{2}{3}\left\langle B^{3/2}\right\rangle \left(\frac{p_{\|a} - p_{\perp a}}{B^{3/2}} - \left\langle \frac{p_{\|a} - p_{\perp a}}{B^{3/2}} \right\rangle\right),$$

$$r_a - \langle r_a \rangle \cong -\frac{2}{3}\left\langle B^{3/2}\right\rangle \left(\frac{r_{\|a} - r_{\perp a}}{B^{3/2}} - \left\langle \frac{r_{\|a} - r_{\perp a}}{B^{3/2}} \right\rangle\right)$$

with an approximation $B^{3/2} \simeq \left\langle B^{3/2}\right\rangle$. Based on this characteristic, the $\left\langle B^{-1}\int_{-1}^{1} P_2(\xi)\left\{\left(V_\| + V_E\right) g_a\right\} \mathrm{d}\xi\right\rangle$ in Eqs.(22)-(23) is considered. The contribution of the operator $V_\|$ in Eq.(22) is negligible since $\left\langle \mathbf{b}\cdot\nabla\left(Q_{\|a}/B\right)\right\rangle$ in this viscosity part is scarcely generated. The $\mathbf{E}\times\mathbf{B}$ operator V_E in Eq.(23) also is negligible because of this local parallel force balance. The integrals $\int v^2 \overline{f}_a \mathrm{d}^3\mathbf{v} \equiv 3p_a/m_a$ and $2\int v^2 P_2(\xi)\overline{f}_a \mathrm{d}^3\mathbf{v} \equiv 2(p_{\|a} - p_{\perp a})/m_a$ will cancel each other there within an accuracy of the approximation $B^{3/2} \simeq \left\langle B^{3/2}\right\rangle$. Therefore, the anisotropy generated in Eq.(25) is a poloidally and toroidally varying one that satisfies $\left\langle p_{\perp a} - p_{\|a}\right\rangle \left\langle (p_{\perp a} - p_{\|a})/B^2\right\rangle < 0$. This characteristic justifies the drift approximation neglecting $|\mathbf{v}_{\mathrm{d}a}\cdot\nabla g_a| \ll \left|\left(V_\| + V_E\right) g_a\right|$, and the collision approximation neglecting $C_{ab}\left(f_{a\mathrm{M}}, g_b^{(l=2)}\right)$ for the calculation that is independent of the anisotropic heating part shown below. On the other hand, the P-S part must handle the separated component $V_\|\left(v_\|\widetilde{U}\right)$ in the radial drift term and the poloidal/toroidal variations of the parallel friction forces. This equation is given by

$$\begin{aligned}
&\left(V_\| + V_E\right) f_{a1}^{\mathrm{PS}} - \sum_{b\neq \mathrm{f}}\left[C_{ab}\left(f_{a1}^{\mathrm{PS}}, f_{b\mathrm{M}}\right) + C_{ab}\left(f_{a\mathrm{M}}, f_{b1}^{\mathrm{PS}}\right)\right] \\
&= \frac{m_a}{\langle T_a\rangle} V_\|\left(v_\|\widetilde{U}\right)\frac{c}{e_a}\left\{X_{a1} - X_{a2}L_1^{(3/2)}(x_a^2)\right\} f_{a\mathrm{M}} \\
&+ C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=1)}\right) - \frac{3}{2}\xi\frac{B}{\langle B^2\rangle}\left\langle B\int_{-1}^{1}\xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=1)}\right)\mathrm{d}\xi\right\rangle.
\end{aligned}$$

Because of the approximation $\int_{-1}^{1}\xi\left(V_E f_{a1}\right)\mathrm{d}\xi = 0$ for the consistency with the MHD equilibrium, this equation is rewritten as

$$\begin{aligned}
&\left(V_\| + V_E\right) g_a^{\mathrm{PS}} - \sum_{b\neq \mathrm{f}}\left[C_{ab}\left(g_a^{\mathrm{PS}}, f_{b\mathrm{M}}\right) + C_{ab}\left(f_{a\mathrm{M}}, g_b^{\mathrm{PS}}\right)\right] \\
&= \widetilde{U}\sum_{b\neq \mathrm{f}}\left[C_{ab}\left(\frac{m_a}{\langle T_a\rangle}v_\|\frac{c}{e_a}\left\{X_{a1} - X_{a2}L_1^{(3/2)}(x_a^2)\right\} f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right)\right. \\
&\qquad \left. + C_{ab}\left(f_{a\mathrm{M}},\ \frac{m_b}{\langle T_b\rangle}v_\|\frac{c}{e_b}\left\{X_{b1} - X_{b2}L_1^{(3/2)}(x_b^2)\right\} f_{b\mathrm{M}}\right)\right] \\
&+ C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=1)}\right) - \frac{3}{2}\xi\frac{B}{\langle B^2\rangle}\left\langle B\int_{-1}^{1}\xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=1)}\right)\mathrm{d}\xi\right\rangle
\end{aligned} \tag{27}$$

for

$$g_a^{\mathrm{PS}} \equiv f_{a1}^{\mathrm{PS}} - \frac{m_a}{\langle T_a \rangle} v_\parallel \widetilde{U} \frac{c}{e_a} \left\{ X_{a1} - X_{a2} L_1^{(3/2)}(x_a^2) \right\} f_{a\mathrm{M}}.$$

The radial electric field $\partial\Phi/\partial s$ in X_{a1} vanishes in the first term $\propto \widetilde{U}$ in the RHS at this step because of the Galilean invariant property of the Coulomb collision. This first term $\propto \widetilde{U}$ will be easily expressed as a Sonine polynomial expansion series using the Braginskii's matrix elements, and the second term corresponding to the friction (momentum exchange) collision with the fast ions also will be calculated as the polynomial expansion series by a method in Sec.IV. When these terms in the RHS are given, the response $g_a^{\mathrm{PS}} = h_a + k_a$ will be determined by

$$\begin{aligned}
&\left(V_\parallel h_a\right)^{(l=0,1)} + V_E^{\mathrm{DKES}} h_a^{(l=0)} - \sum_{b\neq \mathrm{f}} \left[C_{ab}\left(h_a, f_{b\mathrm{M}}\right) + C_{ab}\left(f_{a\mathrm{M}}, h_b\right) \right] \\
&\quad = \widetilde{U} \sum_{b\neq \mathrm{f}} \left[C_{ab}\left(\frac{m_a}{\langle T_a \rangle} v_\parallel \frac{c}{e_a} \left\{ X_{a1} - X_{a2} L_1^{(3/2)}(x_a^2) \right\} f_{a\mathrm{M}},\ f_{b\mathrm{M}} \right) \right. \\
&\qquad \left. + C_{ab}\left(f_{a\mathrm{M}},\ \frac{m_b}{\langle T_b \rangle} v_\parallel \frac{c}{e_b} \left\{ X_{b1} - X_{b2} L_1^{(3/2)}(x_b^2) \right\} f_{b\mathrm{M}} \right) \right] \\
&\quad + C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=1)} \right) - \frac{3}{2} \xi \frac{B}{\langle B^2 \rangle} \left\langle B \int_{-1}^{1} \xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=1)} \right) \mathrm{d}\xi \right\rangle, \\
&\left(V_\parallel + V_E^{\mathrm{DKES}}\right) k_a - \sum_{b\neq \mathrm{f}} \left[C_{ab}\left(k_a, f_{b\mathrm{M}}\right) + C_{ab}\left(f_{a\mathrm{M}}, k_b\right) \right] = - \left(V_\parallel h_a\right)^{(l=2)}
\end{aligned} \tag{28}$$

where the first function h_a includes only the lower Legendre orders $l = 0, 1$ and satisfies $\left\langle B \int_{-1}^{1} \xi h_a \mathrm{d}\xi \right\rangle = 0$, $\left\langle \int_{-1}^{1} h_a \mathrm{d}\xi \right\rangle = 0$. The notation $\left(V_\parallel h_a\right)^{(l)} \equiv \frac{2l+1}{2} P_l(\xi) \int_{-1}^{1} P_l(\xi) \left(V_\parallel h_a\right) \mathrm{d}\xi$ is used. The use of $V_E^{\mathrm{DKES}} h_a^{(l=0)}$ for the solubility condition is justified by this definition in which $\partial h_a^{(l=0)}/\partial v$ in $V_E h_a^{(l=0)}$ becomes a higher order of the **B**-field strength modulation. This problem can be converted to simultaneous algebraic equations for each Fourier modes $(m, n) \neq (0, 0)$ of the Laguerre expansion coefficients $\int L_j^{(1/2)}(x_a^2) h_a \mathrm{d}^3\mathbf{v}$ and $\int v\xi L_j^{(3/2)}(x_a^2) h_a \mathrm{d}^3\mathbf{v}/B$ by taking $\int L_j^{(1/2)}(x_a^2) \mathrm{d}^3\mathbf{v}$ and $\int v\xi L_j^{(3/2)}(x_a^2) \mathrm{d}^3\mathbf{v}$ integrals of this equation. As in Refs.17 and 31, the three terms Laguerre expansion with $j = 0, 1, 2$ will be appropriate. The Fourier expansion in the Boozer coordinates is suited for the ambipolar condition with the finite radial electric field in the $V_E^{\mathrm{DKES}} h_a^{(l=0)}$. Therefore, the function $\widetilde{U}$ also is expressed by using $\varepsilon_{mn}^{(\mathrm{Boozer})} \equiv \frac{1}{2\pi^2} \int_0^{2\pi} \mathrm{d}\theta_{\mathrm{B}} \int_0^{2\pi} \mathrm{d}\zeta_{\mathrm{B}} \left(\langle B^2 \rangle / B^2 - 1 \right) \cos(m\theta_{\mathrm{B}} - n\zeta_{\mathrm{B}})$.[3,9] This is a method for including the field particle portion for these lower Legendre orders $C_{ab}\left(f_{a\mathrm{M}}, h_b^{(l=0,1)}\right)$ and for retaining the momentum/energy conservation. The Onsager symmetric P-S diffusion matrix is obtained at this step since this algebraic method can retain also the self-adjoint relation of the collision operator and the anti-symmetric property $\left\langle \int \widehat{g}_a \left\{ \left(V_\parallel + V_E^{\rm DKES}\right) \widehat{f}_a f_{a\rm M} \right\} {\rm d}^3\mathbf{v} \right\rangle = -\left\langle \int \widehat{f}_a \left\{ \left(V_\parallel + V_E^{\rm DKES}\right) \widehat{g}_a f_{a\rm M} \right\} {\rm d}^3\mathbf{v} \right\rangle$ of the Vlasov operator. After solving these algebraic equations for the Fourier-Legendre-Laguerre series expression of h_a in the 4D space $(\theta_{\rm B}, \zeta_{\rm B}, v, \xi)$, the second equation for the response k_a driven by the source term $-\left(V_\parallel h_a\right)^{(l=2)}$ will be handled by a method that is analogous to the aforementioned procedure for $\sigma_{Xa} \propto P_2(\xi)$ in Eq.(25). The contribution of the $\mathbf{E} \times \mathbf{B}$ being $\int_{-1}^{1} P_2(\xi) \left(V_E^{\rm DKES} h_a\right) {\rm d}\xi = 0$ is not important in the RHS of this equation. An analytical solution given by the Fourier expansion method for the plateau and the P-S collisionality conditions[5,32] $\nu_{\rm D}^a / v > \left| (\delta B/B)^{3/2} \mathbf{b} \cdot \nabla \ln B \right|$ is useful. However, the details of these procedures determining $g_a^{\rm PS} = h_a + k_a$ are beyond the scope of this paper. In particular, the second function k_a is not practically important, since the RHS of Eq.(27) is non-negligible in comparison with that of Eq.(25) only in the P-S collisionality condition, and therefore this function is suppressed by the strong anisotropy relaxation collision in that condition. We shall consider here only a difference between this generation of $-\left(V_\parallel h_a\right)^{(l=2)}$ and σ_{Xa} from the viewpoint of the $\left\langle B^{-1} \int_{-1}^{1} P_2(\xi) \left(V_\parallel \overline{f}_a\right) {\rm d}\xi \right\rangle$ in Eq.(22). When the radial electric field in the $V_E^{\rm DKES} h_a^{(l=0)}$ is finite, the first function h_a includes this $\left\langle \mathbf{b} \cdot \nabla \left(Q_{\parallel a}/B\right) \right\rangle \neq 0$ and thus $\left\langle B^{-1} \int_{-1}^{1} P_2(\xi) k_a {\rm d}\xi \right\rangle \neq 0$ is generated by the balance of the Vlasov and the collision operators. This balance is analogous to the anisotropic heating effect that will be shown in Sec.III-C. But this is only a minor component in k_a that is basically a poloidally and toroidally varying function. By using Eq.(19) again, we find also that $\left\langle B^{3/2} \int_{-1}^{1} P_2(\xi) \left(V_\parallel h_a\right) {\rm d}\xi \right\rangle = 0$. Therefore the k_a in the P-S collisionality condition also is $\left\langle B^J \int_{-1}^{1} P_2(\xi) k_a {\rm d}\xi \right\rangle = 0$ for $J \approx 3/2$. As discussed in Appendix A, for the drift kinetic problems where the Fourier expansion in the Boozer coordinates is essential, we can regard the radial gradient term $(\mathbf{v}_{\rm d a} \cdot \nabla s)\, \partial \left\langle \overline{f}_a \right\rangle / \partial s$ as that of the isotropic-pressure species even when $\left\langle B^2 (p_{\perp a} - p_{\parallel a}) \right\rangle \left\langle p_{\perp a} - p_{\parallel a} \right\rangle < 0$ and $\left\langle B^2 (r_{\perp a} - r_{\parallel a}) \right\rangle \left\langle r_{\perp a} - r_{\parallel a} \right\rangle < 0$. (Instead of the usual surface-averaging for actual geometrical shapes of the flux-surfaces, a simple plane-averaging $\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F {\rm d}\theta_{\rm B} {\rm d}\zeta_{\rm B} = \langle B^2 F \rangle / \langle B^2 \rangle$ for the $(\theta_{\rm B}, \zeta_{\rm B})$ space is used for this judgment on the isotropic-pressure and anisotropic-pressure species.) Therefore the approximation $(\mathbf{v}_{\rm d a} \cdot \nabla s)\, \partial \left\langle \overline{f}_a \right\rangle / \partial s \cong (\mathbf{v}_{\rm d a} \cdot \nabla s)\, \partial f_{a\rm M} / \partial s$ in Eq.(21) is still appropriate even when $\left\langle B^{-1} \int_{-1}^{1} P_2(\xi) k_a {\rm d}\xi \right\rangle \neq 0$ is generated. The velocity distribution component f_{a1} in Eq.(21) generated without the anisotropic heating term is handled by the approximation neglecting $|\mathbf{v}_{\rm d a} \cdot \nabla f_{a1}| \ll \left| \left(V_\parallel + V_E\right) f_{a1} \right|$ for both long-mean-free-path conditions where g_a in Eq.(25) is its dominant component and short-mean-free-path conditions where g_a^{PS} in Eq.(27) is dominant. However, it also should be noted that, for the perpendicular particle/energy fluxes and the resulting classical diffusions $\left\langle \mathbf{\Gamma}_a^{\mathrm{cl}} \cdot \nabla s \right\rangle$ and $\left\langle \mathbf{Q}_a^{\mathrm{cl}} \cdot \nabla s \right\rangle$, this anisotropy generated in the P-S part in Eq.(28) corresponds to a finite contribution of $\langle B \rangle \frac{\partial}{\partial s} \left\langle \int v^k P_2(\xi) k_a \mathrm{d}^3\mathbf{v}/B^3 \right\rangle$ with $k = 2, 4$ in $(\nabla \cdot \boldsymbol{\pi}_a)_\perp / B^2 = \frac{1}{3} B \nabla_\perp \left\{ (p_{\perp a} - p_{\| a})/B^3 \right\}$ and $\left\{ \nabla \cdot (\mathbf{r}_a - r_a \mathbf{I}) \right\}_\perp / B^2 = \frac{1}{3} B \nabla_\perp \left\{ (r_{\perp a} - r_{\| a})/B^3 \right\}$.

In addition to these viscosity and the P-S parts for handling $\mathbf{v}_{\mathrm{d}a} \cdot \nabla s$, $E_\|^{(\mathbf{A})}$, and $C_{a\mathrm{f}} \left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=1)} \right)$ in the RHS in Eq.(21), we must investigate also the response to the anisotropic heating term $C_{a\mathrm{f}} \left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=2)} \right)$. As already mentioned, however, this newly generated velocity distribution component has contrasting characteristics. One is that the assumption $|\mathbf{v}_{\mathrm{d}a} \cdot \nabla f_{a1}| \ll \left| \left(V_\| + V_E \right) f_{a1} \right|$ is not appropriate for this component. The second is that the first Legendre order $f_{a1}^{(l=1)}$ and the higher Legendre orders $f_{a1}^{(l \geq 3)}$ are scarcely generated by this source term. Therefore, after the determination including the $C_{ab} \left(f_{a\mathrm{M}}, f_{b1}^{(l=2)} \right)$ (except the small mass ratio approximation for the electron-ion and the ion-electron collisions), we may need to include the obtained anisotropy in the radial gradients $\partial \left\langle p_{\perp a} + p_{\| a} \right\rangle / \partial s$, $\partial \left\langle r_{\perp a} + r_{\| a} \right\rangle / \partial s$ in Eqs.(A14-A15) and the corresponding DKE term $(\mathbf{v}_{\mathrm{d}a} \cdot \nabla s) \partial f_{a\mathrm{M}} / \partial s$ in Eq.(21). In this study, however, we still assume the nearly thermalized states for which $\left\langle \mathbf{\Gamma}_a^{\mathrm{bn}} \cdot \nabla s \right\rangle$, $\left\langle \mathbf{q}_a^{\mathrm{bn}} \cdot \nabla s \right\rangle$, $\langle \mathbf{B} \cdot \mathbf{J} \rangle - e_{\mathrm{f}} \left\langle \mathbf{B} \cdot \int \mathbf{v} f_{\mathrm{f}} \mathrm{d}^3\mathbf{v} \right\rangle$, $\left\langle \mathbf{\Gamma}_a^{\mathrm{PS}} \cdot \nabla s \right\rangle$, and $\left\langle \mathbf{q}_a^{\mathrm{PS}} \cdot \nabla s \right\rangle$ can be expressed by Onsager symmetric transport matrices as in Refs.3–6 and Ref.33. This symmetry requires the (θ, ζ, v, ξ) space structure of the radial gradient term $(\mathbf{v}_{\mathrm{d}a} \cdot \nabla s) \partial \left\langle f_a^{(\mathrm{even})} \right\rangle / \partial s \propto v^2 (1 + \xi^2) \left\{ 1 + \alpha L_1^{(3/2)}(x_a^2) \right\} f_{a\mathrm{M}} \nabla s \times \mathbf{B} \cdot \nabla B^{-2}$ even when absolute values of $\partial \langle p_a \rangle / \partial s$ and $\partial \langle r_a \rangle / \partial s$ in it are modified. There are two reasons for which we do not consider this matrix expression for the radial and the parallel particle/energy fluxes of the NB-produced fast ions. One is that the self-adjoint property of the collision does not exist there, and the second is that their radial gradient $\partial \left\langle f_{\mathrm{f}}^{(\mathrm{even})} \right\rangle / \partial s$ does not have this $\mathbf{v}$-space structure. In contrast to this, we assume nearly isotropic states $\left| \left\langle f_a^{(l=2)} / B \right\rangle / \langle B^{-1} \rangle \right| \ll f_{a\mathrm{M}}$ for thermalized particle species and thus the above $\mathbf{v}$-space structure of the radial gradient term is used. One purpose of this section is to show a method for confirming the validity of this assumption for each experimental condition. The basic (θ, ζ, v, ξ) space structure of the response to the new source term $C_{a\mathrm{f}} \left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=2)} \right)$ is the aforementioned

$$g_a^{\mathrm{an}} \cong \frac{1}{\langle B^{-2}\rangle}\left(\frac{P_2(\xi)}{B}-\frac{1}{B}+\left\langle\frac{1}{B}\right\rangle\right)\frac{m_a v^2}{3\langle T_a\rangle}f_{a\mathrm{M}}\sum_{j=0}^{\infty}\left\langle\frac{p_{2aj}}{B}\right\rangle L_j^{(5/2)}(x_a^2) \ll f_{a\mathrm{M}} \tag{29}$$

that is expressed by the series of the Sonine polynomial $L_j^{(5/2)}(x_a^2)$. Although we shall consider only this basic structure in the DKEs for thermalized ions ($a \neq$ e,f), a small deviation from this form will be included in the electron velocity distribution function ($a=$ e) as discussed below. The operator $V_\|$ vanishes for this basic structure. The contributions of the $l=0$ energy scattering/exchange collision effect $\int_{-1}^{1} C_{ab}\left(\overline{f}_a, \overline{f}_b\right) \mathrm{d}\xi$ and $c\nabla\Phi\times\mathbf{B}\cdot\nabla f_a^{(l=0)}$ in the V_E operator for the lowest Legendre order component $\propto -1/B+\langle 1/B\rangle$ to the parallel flow divergence $\mathbf{B}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_a \mathrm{d}\xi/B\right)$ and the resultant parallel friction collision can be neglected analogously to the viscosity part Eq.(25). Therefore, the equation

$$\left(V_\| + V_E\right) g_a^{\mathrm{an}} - \sum_{b\neq\mathrm{f}}\left[C_{ab}\left(g_a^{\mathrm{an}}, f_{b\mathrm{M}}\right) + C_{ab}\left(f_{a\mathrm{M}}, g_b^{\mathrm{an}}\right)\right] = C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l=2)}\right) \tag{30}$$

for this determination is independent of the P-S part Eq.(27). However, V_E for the dominant component $P_2(\xi)$ must be that in Eq.(6). This $V_E g_a^{\mathrm{an}}$ will have a role that is analogous to $V_E f_{a\mathrm{M}}$ in the RHS in Eq.(21). Here, we shall investigate this effect for the lowest Laguerre order term $j=0$ (corresponding to the pressure anisotropy) $P_2(\xi)v^2 f_{a\mathrm{M}}/B$. It is given by

$$\begin{aligned} & V_E\left(\frac{P_2(\xi)}{B}v^2 f_{a\mathrm{M}}\right) \\ &= cE_s\left(\frac{\nabla s\times\mathbf{B}}{B^3}\cdot\nabla\ln B\right)v^2\left\{-\frac{12}{35}x_a^2 P_4(\xi)+\frac{2}{3}\left(1-\frac{16}{7}x_a^2\right)P_2(\xi)+\frac{2}{15}\left(\frac{5}{2}-x_a^2\right)\right\}f_{a\mathrm{M}}. \end{aligned}$$

When this type of source term is included in the RHS of Eq.(21), the solving procedure will be analogous to the $\mathbf{v}_{\mathrm{d}a}\cdot\nabla s$ in the viscosity and the P-S parts in Eqs.(25) and (27), and analogous velocity distribution components will be generated there. However, this generation is not important as long as the nearly isotropic condition $|g_a^{\mathrm{an}}| \ll f_{a\mathrm{M}}$ is satisfied. As shown in Eqs.(A14-A15), this contribution of the anisotropy to the divergences of the $\mathbf{E}\times\mathbf{B}$ particle/energy fluxes $\propto \nabla\Phi\times\mathbf{B}\cdot\nabla B^{-2}$ is small compared with the divergences of the diamagnetic perpendicular particle/energy fluxes, and is only to change $5\langle p_a\rangle E_s\nabla s\times\mathbf{B}\cdot\nabla B^{-2}$ to $\left\langle 3p_a+p_{\perp a}+p_{\|a}\right\rangle E_s\nabla s\times\mathbf{B}\cdot\nabla B^{-2}$ in $\mathbf{B}\cdot\nabla\left(Q_{\|a}/B\right)$. This fact can be confirmed also by integrations by parts for Eq.(6), and is unchanged even when the higher Laguerre orders $j\geq 1$ in Eq.(29) are included. Therefore, we shall neglect $V_E g_a^{\mathrm{an}}$ in comparison with $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\left\{X_{a1}-X_{a2}L_1^{(3/2)}(x_a^2)\right\} f_{a\mathrm{M}}/\langle T_a\rangle$. These approximations are commonly used for the thermalized ions and the electrons. One remaining issue is the small deviation of the electron solution from the basic form in Eq.(29). This deviation is caused by the fact that the real space structure of the electron heating term $C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_{\mathrm{f}}^{(l=2)}\right)$ will not have the form of $\propto P_2(\xi)/B(\theta,\zeta)$ as mentioned at the end of Sec.II. Rather than this form, this term may be close to $C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_{\mathrm{f}}^{(l=2)}\right) \simeq \frac{5}{2}P_2(\xi)\left\langle B^{-1}\int_{-1}^{1} P_2(\xi)C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, \overline{f}_{\mathrm{f}}\right)\mathrm{d}\xi\right\rangle B$. Since this problem is independent of the anisotropic heating of thermalized ions because of $C_{\mathrm{ei}}\left(f_{\mathrm{eM}}, f_{\mathrm{i1}}^{(l\geq 2)}\right) = 0$ in the usual small mass ratio approximation, we should solve only

$$-\left[\sum_{b\neq \mathrm{f}} C_{\mathrm{e}b}\left(g_{\mathrm{e0}}^{\mathrm{an}}, f_{b\mathrm{M}}\right) + C_{\mathrm{ee}}\left(f_{\mathrm{eM}}, g_{\mathrm{e0}}^{\mathrm{an}}\right)\right] = C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_{\mathrm{f}}^{(l=2)}\right) \tag{31}$$

and

$$V_{\parallel} g_{\mathrm{e1}}^{\mathrm{an}} - \sum_{b\neq \mathrm{f}} C_{\mathrm{e}b}\left(\left(g_{\mathrm{e1}}^{\mathrm{an}}\right)^{(l\neq 0)}, f_{b\mathrm{M}}\right) = -V_{\parallel} g_{\mathrm{e0}}^{\mathrm{an}}. \tag{32}$$

When the anisotropic heating source term has a form of $C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_{\mathrm{f}}^{(l=2)}\right) = f\left(v\right)P_2(\xi)F\left(\theta,\zeta\right)$ on each flux-surface in the first equation, its solution also has the same real space structure and thus $g_{\mathrm{e0}}^{\mathrm{an}} = g\left(v\right)P_2(\xi)F\left(\theta,\zeta\right)$. This problem is solved by using the anisotropy relaxation matrix in Appendix B. In Eq.(32) for a small component $g_{\mathrm{e1}}^{\mathrm{an}}$, the $V_E g_{\mathrm{e0}}^{\mathrm{an}}$ being smaller than $\left(\mathbf{v}_{\mathrm{de}}\cdot\nabla s\right)\left\{X_{\mathrm{e1}} - X_{\mathrm{e2}}L_1^{(3/2)}(x_{\mathrm{e}}^2)\right\} f_{\mathrm{eM}}/\left\langle T_{\mathrm{e}}\right\rangle$ is neglected in the RHS as already stated, and the approximation neglecting $\left|\left(V_E + \mathbf{v}_{\mathrm{de}}\cdot\nabla\right) g_{\mathrm{e1}}^{\mathrm{an}}\right| \ll \left|V_{\parallel} g_{\mathrm{e1}}^{\mathrm{an}}\right|$ in the LHS is appropriate because of the large thermal velocity $\sqrt{2T_{\mathrm{e}}/m_{\mathrm{e}}}$. For the collision operator for this $g_{\mathrm{e1}}^{\mathrm{an}}$, we shall adopt the method for Eq.(25) (only $\nu_{\mathrm{D}}^{\mathrm{e}}(v)\mathcal{L}g_{\mathrm{e1}}^{\mathrm{an}}$ substantially). The purpose of this investigation is only to understand that the deviation of $g_{\mathrm{e}}^{\mathrm{an}} = g_{\mathrm{e0}}^{\mathrm{an}} + g_{\mathrm{e1}}^{\mathrm{an}}$ from the form in Eq.(29) includes only the poloidal/toroidal variations of the even function of ξ with the symmetric phase ($P_l(\xi)\cos\left(m\theta - n\zeta\right)$ with $l = 0, 2, 4, ...$) and those of the odd function with the anti-symmetric phase ($P_l(\xi)\sin\left(m\theta - n\zeta\right)$ with $l = 3, 5, ...$) and that these velocity distribution components are irrelative to $\left\langle \boldsymbol{\Gamma}_a^{\mathrm{bn}}\cdot\nabla s\right\rangle$, $\left\langle \mathbf{q}_a^{\mathrm{bn}}\cdot\nabla s\right\rangle$, $\left\langle \mathbf{B}\cdot\mathbf{J}\right\rangle - e_{\mathrm{f}}\left\langle \mathbf{B}\cdot\int \mathbf{v} f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\right\rangle$, $\left\langle \boldsymbol{\Gamma}_a^{\mathrm{PS}}\cdot\nabla s\right\rangle$, and $\left\langle \mathbf{q}_a^{\mathrm{PS}}\cdot\nabla s\right\rangle$. Analogous to the analytical methods for solving Eq.(26),[18] two methods for the two v-space regions of the second function $g_{\mathrm{e1}}^{\mathrm{an}}$ are considered. The first method is that for the plateau and P-S energy region $\nu_{\mathrm{D}}^{\mathrm{e}}/v > \left|(\delta B/B)^{3/2}\mathbf{b}\cdot\nabla\ln B\right|$ using the Fourier expansion for the (θ,ζ,ξ) expressions of the solution and the DKE terms, and the second is that for the long-mean-free-path (banana) energy region using the (θ,ζ,λ) expression that is analogous to Sec.II for the fast ions. Firstly, the plateau and the P-S energy region is calculated by using Eq.(19) for the source term $-V_{\|}g^{\mathrm{an}}_{\mathrm{e0}} = -g\left(v\right)V_{\|}\left\{P_2(\xi)F\left(\theta,\zeta\right)\right\}$ in Eq.(32) as follows:

$$V_{\|}\left\{P_2(\xi)F\left(\theta,\zeta\right)\right\}$$
$$= v\xi\left(\frac{\langle F/B\rangle}{\langle B^{-2}\rangle}\mathbf{b}\cdot\nabla\frac{1}{B}+\frac{2}{5}B^{3/2}\mathbf{b}\cdot\nabla\frac{F-\langle F/B\rangle B^{-1}/\langle B^{-2}\rangle}{B^{3/2}}\right)+\frac{3}{5}vP_3(\xi)\frac{1}{B}\mathbf{b}\cdot\nabla\left(FB\right). \tag{33}$$

By the "frictionless"local parallel force balance, this first term $\propto v\xi$ generates a velocity distribution component being the lowest Legendre order $l = 0$

$$f^0 =$$
$$-\frac{\langle F/B\rangle}{\langle B^{-2}\rangle}\left(\frac{1}{B}-\left\langle\frac{1}{B}\right\rangle\right)-\frac{2}{5}\left\langle B^{3/2}\right\rangle\left\{\frac{F}{B^{3/2}}-\left\langle\frac{F}{B^{3/2}}\right\rangle-\frac{\langle F/B\rangle}{\langle B^{-2}\rangle}\left(\frac{1}{B^{5/2}}-\left\langle\frac{1}{B^{5/2}}\right\rangle\right)\right\} \tag{34}$$

in the response $g^{\mathrm{an}}_{\mathrm{e1}}/g(v)$. Here, an approximation $B^{3/2} \cong \left\langle B^{3/2}\right\rangle$ is used for the minor components. When $F \propto B^{-1}$, this second term vanishes. The response to the second term $\propto vP_3(\xi)$ in Eq.(33) is obtained by using the Fourier expansions of

$$\begin{aligned}\frac{1}{B}\mathbf{b}\cdot\nabla\left(FB\right) &= \frac{1}{\langle B^2\rangle}\frac{4\pi^2}{V'}\left(\chi'\frac{\partial}{\partial\theta_{\mathrm{B}}}+\psi'\frac{\partial}{\partial\zeta_{\mathrm{B}}}\right)\left(FB\right)\\ FB &= \sum_{m,n}\left(FB\right)_{mn}\cos(m\theta_{\mathrm{B}}-n\zeta_{\mathrm{B}})\end{aligned} \tag{35}$$

and the solution

$$g^{\mathrm{an}}_{\mathrm{e1}}/g\left(v\right)-f^0 = \sum_{(m,n)\neq(0,0)} f^{\mathrm{c}}_{mn}(v,\xi)\cos(m\theta_{\mathrm{B}}-n\zeta_{\mathrm{B}})+\sum_{(m,n)\neq(0,0)} f^{\mathrm{s}}_{mn}\left(v,\xi\right)\sin(m\theta_{\mathrm{B}}-n\zeta_{\mathrm{B}}) \tag{36}$$

in the Boozer coordinates, and an approximation of the operators $V_{\|}-\nu^{\mathrm{e}}_{\mathrm{D}}\mathcal{L} \simeq \left(\langle B\rangle/B\right)v\xi\mathbf{b}\cdot\nabla_{(v,\xi)=\mathrm{const}}+6\nu^{\mathrm{e}}_{\mathrm{D}}$ in the LHS of Eq.(32). The electron Krook collision operator is chosen to be $-6\nu^{\mathrm{e}}_{\mathrm{D}}$ for the polynomial $P_3(\xi)$. The result of this method for the short-mean-free-path energy region is

$$\begin{aligned}f^{\mathrm{s}}_{mn} &= \frac{3}{5}\frac{vP_3(\xi)}{\langle B^2\rangle}\frac{4\pi^2}{V'}\frac{6\nu^{\mathrm{e}}_{\mathrm{D}}}{\left\{\left(v\xi/\langle B\rangle\right)\left(4\pi^2/V'\right)\left(\chi'm-\psi'n\right)\right\}^2+\left(6\nu^{\mathrm{e}}_{\mathrm{D}}\right)^2}\left(\chi'm-\psi'n\right)\left(FB\right)_{mn}\\ f^{\mathrm{c}}_{mn} &= -\frac{3}{5}\frac{vP_3(\xi)}{\langle B^2\rangle}\frac{4\pi^2}{V'}\frac{\left(v\xi/\langle B\rangle\right)\left(4\pi^2/V'\right)\left(\chi'm-\psi'n\right)}{\left\{\left(v\xi/\langle B\rangle\right)\left(4\pi^2/V'\right)\left(\chi'm-\psi'n\right)\right\}^2+\left(6\nu^{\mathrm{e}}_{\mathrm{D}}\right)^2}\left(\chi'm-\psi'n\right)\left(FB\right)_{mn}.\end{aligned} \tag{37}$$

Because of a rough approximation in the LHS of the equation, this solution cannot be used for the purpose of $\int_{-1}^{1}\xi f^{\mathrm{s}}_{mn}\mathrm{d}\xi$ that does not exist actually due to the particle/energy conservation. This solution cannot be used also for $\int_{-1}^{1} f_{mn}^{\mathrm{c}}\mathrm{d}\xi$ because of this violation of $\int_{-1}^{1}\xi f_{mn}^{\mathrm{s}}\mathrm{d}\xi = 0$. The "frictionless"local parallel force balance will determine the correct value of this lowest Legendre order. The method for the long-mean-free-path energy region is analogous to that in Sec.II, and is easier since we should calculate only the PAS operator $\mathcal{L}$. The solution will have the form

$$g_{\mathrm{e1}}^{\mathrm{an}}/g\left(v\right) = -P_2(\xi)F\left(\theta,\zeta\right) + g_2(\lambda), \tag{38}$$

and we determine the integration constant $g_2(\lambda)$ by the solubility condition for the next order of $\nu_{\mathrm{D}}^{\mathrm{e}}/v$. For the circulating pitch-angle range $\lambda \leq 1$, this calculation is

$$\begin{aligned} \left\langle \frac{B}{v_\parallel}\mathcal{L}\left\{-P_2(\xi)F\left(\theta,\zeta\right) + g_2(\lambda)\right\}\right\rangle = 0, \\ \frac{\partial g_2}{\partial\lambda} = -\frac{3}{2}\left\langle\frac{v_\parallel}{v}\right\rangle^{-1}\left\langle\frac{B}{B_{\mathrm{M}}}\frac{v_\parallel}{v}F\left(\theta,\zeta\right)\right\rangle \simeq -\frac{3}{2}\left\langle\frac{B}{B_{\mathrm{M}}}F\left(\theta,\zeta\right)\right\rangle. \end{aligned} \tag{39}$$

Analogous to the calculation in Sec.II, this $g_2(\lambda)$ exists in the full pitch angle range $0 \leq \lambda \leq B_{\mathrm{M}}/B$, and the trapped range $\lambda > 1$ is determined by the bounce integral

$$\frac{\partial g_2}{\partial\lambda} = -\frac{3}{2}\left(\oint\frac{v_\parallel}{v}\mathrm{d}l\right)^{-1}\oint\left\{\frac{B}{B_{\mathrm{M}}}\frac{v_\parallel}{v}F(\theta,\zeta)\right\}\mathrm{d}l \simeq -\frac{3}{2}\left(\oint\mathrm{d}l\right)^{-1}\oint\left\{\frac{B}{B_{\mathrm{M}}}F(\theta,\zeta)\right\}\mathrm{d}l \tag{40}$$

instead of the surface-averaging. This $\partial g_2/\partial\lambda$ is continuous at trapped/circulating boundary $\lambda = 1$. Since $B(\theta,\zeta)F(\theta,\zeta)$ is a moderately varying function on the flux-surface in comparison with the parallel particle velocity $v_\parallel$, this solution $g_2(\lambda)$ is almost a linear function of λ in the full pitch-angle range. The boundary condition of this $\int\mathrm{d}\lambda$ integral is determined by $\left\langle\int_{-1}^{1} g_2\mathrm{d}\xi\right\rangle = 0$ for the full (θ,ζ,λ) range corresponding to the definition of $\langle n_{\mathrm{e}}\rangle$ and $\langle p_{\mathrm{e}}\rangle$. Therefore, this function is roughly estimated to be $g_2 \simeq \langle F(\mathbf{x},v)/B\rangle\left(P_2(\xi)/B - 1/B + \langle 1/B\rangle\right)/\langle B^{-2}\rangle$. Therefore the solution $g_{\mathrm{e}}^{\mathrm{an}} = g_{\mathrm{e0}}^{\mathrm{an}} + g_{\mathrm{e1}}^{\mathrm{an}}$ in the long-mean-free-path energy region becomes insensitive to the deviation of the anisotropic electron heating term $C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_{\mathrm{f}}^{(l=2)}\right)$ from the form of $\propto P_2(\xi)/B\left(\theta,\zeta\right)$ that was mentioned in Sec.II. In conclusion on the newly added part Eq.(30) (both the thermalized ions and the electrons), this part is irrelative to the previous obtaining procedures for the transport matrix elements and can be calculated independently even if its result may become minor corrections of the radial gradients $\partial\left\langle p_{\perp a} + p_{\parallel a}\right\rangle/\partial s$, $\partial\left\langle r_{\perp a} + r_{\parallel a}\right\rangle/\partial s$ as the thermodynamic forces in this matrix expression.

C. Solution of the anisotropic heating part

Since Eq.(30) with the neglect of $V_E g_a^{\mathrm{an}}$ is a integro-differential equation including the integral operator $C_{ab}\left(f_{a\mathrm{M}}, g_b^{\mathrm{an}}\right)$ of like-particle collisions $a = b$ and of unlike-ion collision $a \neq b$, we shall convert it to the simultaneous algebraic equations for obtaining $\langle p_{2aj}/B\rangle \equiv \frac{15\cdot 2^j j!}{(2j+5)!!} m_a \left\langle \int v^2 P_2(\xi) L_j^{(5/2)}(x_a^2) f_a \mathrm{d}^3\mathbf{v}/B \right\rangle / \langle p_a\rangle$ by taking the Laguerre expansion moments of

$$-\sum_{b\neq \mathrm{f}} \left\langle B^{-1} \int_{-1}^{1} P_2(\xi) \left[C_{ab}\left(g_a^{\mathrm{an}}, f_{b\mathrm{M}}\right) + C_{ab}\left(f_{a\mathrm{M}}, g_b^{\mathrm{an}}\right)\right] \mathrm{d}\xi \right\rangle \\ = \left\langle B^{-1} \int_{-1}^{1} P_2(\xi) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, \overline{f}_{\mathrm{f}}\right) \mathrm{d}\xi \right\rangle, \tag{41}$$

and using the Braginskii's matrix expression of the full linearized collision operator in Appendix B. One contrasting situation, which is quite different from the previous determination of the first Legendre order components $\left\langle Bu_{\|aj}\right\rangle \equiv \frac{3\cdot 2^j j!}{(2j+3)!!} \left\langle B \int v\xi L_j^{(3/2)}(x_a^2) f_a \mathrm{d}^3\mathbf{v} \right\rangle / \langle n_a\rangle$, is that we eliminated the operator $V_\|$ even for the electrons by taking the $\left\langle B^{-1} \int_{-1}^{1} P_2(\xi) \mathrm{d}\xi \right\rangle$ integral of the DKE. In the determination of the $\left\langle Bu_{\|aj}\right\rangle$, the parallel viscosity matrix $M^a_{j+1,k+1}$ for these parallel flow moments that expresses a part of $\left\langle B \int_{-1}^{1} \xi \left(V_\| f_a\right) \mathrm{d}\xi \right\rangle$ was required together with the friction matrix $l^{ab}_{j+1,k+1}$ since the latter matrix does not have its inverse matrix because of the momentum conserving and Galilean invariant property of the Coulomb collision.[4–6] In contrast to this, the contribution of the $V_\|$ operator is not essential for the determination of the $\langle p_{2aj}/B\rangle$ integrals. Therefore, we should solve only a simple algebraic equation

$$-\begin{bmatrix} \mathbf{Q}^{aa} & \mathbf{Q}^{ab} & \mathbf{Q}^{ac} & \cdot\ \cdot & \mathbf{Q}^{aN} \\ \mathbf{Q}^{ba} & \mathbf{Q}^{bb} & \mathbf{Q}^{bc} & \cdot\ \cdot & \mathbf{Q}^{bN} \\ \mathbf{Q}^{ca} & \mathbf{Q}^{cb} & \mathbf{Q}^{cc} & \cdot\ \cdot & \mathbf{Q}^{cN} \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \mathbf{Q}^{Na} & \mathbf{Q}^{Nb} & \mathbf{Q}^{Nc} & \cdot\ \cdot & \mathbf{Q}^{NN} \end{bmatrix} \begin{bmatrix} \mathbf{P}_a \\ \mathbf{P}_b \\ \mathbf{P}_c \\ \cdot \\ \cdot \\ \mathbf{P}_N \end{bmatrix} = \begin{bmatrix} \mathbf{C}_a \\ \mathbf{C}_b \\ \mathbf{C}_c \\ \cdot \\ \cdot \\ \mathbf{C}_N \end{bmatrix}. \tag{42}$$

Here, $\mathbf{Q}^{ab}$ are matrices of the Braginskii's elements in Appendix B ($\mathbf{Q}^{eb} = \mathbf{Q}^{be} = 0$ for $b \neq e$), $\mathbf{P}_a$ is a vector of $\frac{2}{3}\langle p_{2aj}/B\rangle$, and $\mathbf{C}_a$ is a vector of $\left\langle B^{-1} \int x_a^2 P_2(\xi) L_j^{(5/2)}(x_a^2) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right) \mathrm{d}^3\mathbf{v} \right\rangle$ of the thermalized particle species a. In a previous investigation on the parallel flow moments $\left\langle Bu_{\|aj}\right\rangle$, it was found that the Laguerre expansion with the three terms $j = 0, 1, 2$

is sufficiently accurate for various conditions of various thermalized particle species.[5,6] Following this experience, the three terms expansion is used also for this anisotropic heating analysis. As pointed out on the P-S part in Eq.(28) with $E_s \neq 0$, the Laguerre expansion coefficients of $\left\langle B^{-1}\int_{-1}^{1} P_2(\xi)\left\{\left(V_\| + V_E\right)h_a\right\}\mathrm{d}\xi\right\rangle$ also can be included in this RHS as the source term. However, the response to this source term component due to Eq.(28) will not substantially change the radial gradients $\partial\left\langle p_{\perp a} + p_{\| a}\right\rangle/\partial s$, $\partial\left\langle r_{\perp a} + r_{\| a}\right\rangle/\partial s$ in the DKE for determining the gyro-phase-averaged velocity distribution, even if it may be a part of a finite contribution $\langle B\rangle \frac{\partial}{\partial s}\left\langle\int v^k P_2(\xi) k_a \mathrm{d}^3\mathbf{v}/B^3\right\rangle$ in the perpendicular particle/energy fluxes and the resultant classical diffusions. Here, we investigate only the response to $\left\langle B^{-1}\int x_a^2 P_2(\xi) L_j^{(5/2)}(x_a^2) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_\mathrm{f}\right)\mathrm{d}^3\mathbf{v}\right\rangle$ for the gyro-phase-averaged velocity distribution. This integral of the anisotropic heating source term is calculated by the formulas[19]

$$\int x_a^2 P_2(\xi) C_{ab}\left(f_{aM}, f_b\right)\mathrm{d}^3\mathbf{v} = 16\pi^2 n_a\left(\frac{e_a e_b}{m_a}\right)^2 \ln\Lambda_{ab}$$
$$\times\int_0^\infty\left[\left(\frac{m_a}{m_b}+\frac{3}{2}\right)\left\{\frac{3G(x_a)}{x_a}-\frac{2}{\sqrt{\pi}}\exp\left(-x_a^2\right)\right\}-x_a G(x_a)\right]\left(\int_{-1}^{1}P_2(\xi)\overline{f}_b \mathrm{d}\xi\right)x_a^2\mathrm{d}x_a, \tag{43}$$

$$\int x_a^2 L_1^{(5/2)}\left(x_a^2\right) P_2(\xi) C_{ab}\left(f_{aM}, f_b\right)\mathrm{d}^3\mathbf{v} = 16\pi^2 n_a\left(\frac{e_a e_b}{m_a}\right)^2 \ln\Lambda_{ab}$$
$$\times\int_0^\infty\left[4\left(1+\frac{m_a}{m_b}\right)\frac{x_a^2}{\sqrt{\pi}}\exp\left(-x_a^2\right)-3\left\{\frac{3G(x_a)}{x_a}-\frac{2}{\sqrt{\pi}}\exp\left(-x_a^2\right)\right\}\right]\left(\int_{-1}^{1}P_2(\xi)\overline{f}_b \mathrm{d}\xi\right)x_a^2\mathrm{d}x_a, \tag{44}$$

and

$$\int x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) C_{ab}\left(f_{a\mathrm{M}}, f_b\right)\mathrm{d}^3\mathbf{v}$$
$$= 24\pi^{3/2} n_a\left(\frac{e_a e_b}{m_a}\right)^2 \ln\Lambda_{ab}\int_0^\infty\left\{7\frac{m_a}{m_b}+2-2\left(\frac{m_a}{m_b}+1\right)x_a^2\right\}\exp\left(-x_a^2\right)\left(\int_{-1}^{1}P_2(\xi)\overline{f}_b \mathrm{d}\xi\right)x_a^4\mathrm{d}x_a. \tag{45}$$

Here, a function

$$3G(x) - \frac{2}{\sqrt{\pi}}x\exp\left(-x^2\right) = \frac{4}{\sqrt{\pi}}x^{-2}\int_0^x y^4\exp\left(-y^2\right)dy = \frac{4}{\sqrt{\pi}}x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+5)n!}x^{2n}$$

is included in the Laguerre orders $j = 0, 1$ in Eqs.(43)-(44). This function will often appear also in Sec.IV. We shall use Eqs.(5) and (16) for numerical integrals of these $\int x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_\mathrm{f}\right)\mathrm{d}^3\mathbf{v}$ formulas. For the boundary condition at $v = 0$ assumed in this adjoint equation method, straightforward use of these formulas is favorable. Although the $\int x_\mathrm{e}^2 L_j^{(5/2)}(x_\mathrm{e}^2) P_2(\xi) C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_\mathrm{f}\right)\mathrm{d}^3\mathbf{v}$ integrals of the e-f collision in injection conditions with $v_{\mathrm{b}}^2 < 2T_{\mathrm{e}}/m_{\mathrm{e}}$ may be obtained also by using $\left\langle \int v^k P_2(\xi) f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}/B \right\rangle$ with $k \geq 2$ that will be discussed in Sec.IV, this method needs appropriate polynomial expressions of $G(x)$, $3G(x) - \frac{2}{\sqrt{\pi}} x\exp(-x^2)$, and $\exp(-x^2)$ that depend on the injection velocity v_{b}. The straightforward numerical integral is a method for handling arbitrary v_{b}. For a qualitative understanding, however, the following approximations for $v_{\mathrm{b}}^2 \ll 2T_{\mathrm{e}}/m_{\mathrm{e}}$ also are useful:

$$
\begin{aligned}
\int x_{\mathrm{e}}^2 P_2(\xi) C_{\mathrm{ef}}(f_{\mathrm{eM}}, f_{\mathrm{f}}) \, \mathrm{d}^3\mathbf{v} &\simeq \frac{2}{5\tau_{\mathrm{S}}} \frac{p_{\parallel\mathrm{f}} - p_{\perp\mathrm{f}}}{\langle T_{\mathrm{e}} \rangle}, \\
\int x_{\mathrm{e}}^2 L_1^{(5/2)}(x_{\mathrm{e}}^2) P_2(\xi) C_{\mathrm{ef}}(f_{\mathrm{eM}}, f_{\mathrm{f}}) \, \mathrm{d}^3\mathbf{v} &\simeq \frac{6}{5\tau_{\mathrm{S}}} \frac{p_{\parallel\mathrm{f}} - p_{\perp\mathrm{f}}}{\langle T_{\mathrm{e}} \rangle}, \\
\int x_{\mathrm{e}}^2 L_2^{(5/2)}(x_{\mathrm{e}}^2) P_2(\xi) C_{\mathrm{ef}}(f_{\mathrm{eM}}, f_{\mathrm{f}}) \, \mathrm{d}^3\mathbf{v} &\simeq \frac{9}{4\tau_{\mathrm{S}}} \frac{p_{\parallel\mathrm{f}} - p_{\perp\mathrm{f}}}{\langle T_{\mathrm{e}} \rangle}.
\end{aligned}
\tag{46}
$$

Here, $\tau_{\mathrm{S}} \equiv 3m_{\mathrm{f}} m_{\mathrm{e}} v_{\mathrm{Te}}^3 / \left(16\sqrt{\pi} e^4 Z_{\mathrm{f}}^2 \langle n_{\mathrm{e}} \rangle \ln\Lambda_{\mathrm{fe}}\right)$ is the slowing down time that is already used in Eq.(2). The dependence of these $\int x_{\mathrm{e}}^2 L_j^{(5/2)}(x_{\mathrm{e}}^2) P_2(\xi) C_{\mathrm{ef}}(f_{\mathrm{eM}}, f_{\mathrm{f}}) \, \mathrm{d}^3\mathbf{v}$ integrals on the $\mathbf{B}$-field strength modulation discussed in Sec.II is close to that of $\left\langle (p_{\parallel\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle$. This dependence will be shown in Sec.IV. In this section, only the dependence of $\int x_a^2 L_j^{(5/2)}(x_a^2) P_2(\xi) C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}}) \, \mathrm{d}^3\mathbf{v}$ with $j = 0, 1$ of ions ($a \neq$ e,f) is shown in Fig.2. As understood by Eq.(12), this $j = 1$ integral is determined by a velocity range $0 \leq v \lesssim v_{\mathrm{c}}$ of the slowing down velocity distribution and must be included as a typical application example of Eqs.(5) and (16) of the adjoint equation method. On the other hand, $\int x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}}) \, \mathrm{d}^3\mathbf{v}$ is only a contribution of the thermal velocity region $v^2 \sim 2T_{\mathrm{i}}/m_a$ and thus is negligibly smaller than the $j = 0, 1$ integrals. Since various approximations in Eqs.(1)-(2) become physically meaningless for this thermal velocity region, this $j = 2$ integral is not shown. In Fig.2, these results of the adjoint equation method are normalized by $\int x_a^2 L_j^{(5/2)}(x_a^2) P_2(\xi) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, \overline{f}_{\mathrm{f}}^{(\mathbf{b}\cdot\nabla B=0)}\right) \mathrm{d}^3\mathbf{v}$ that are given by Eq.(20). The assumed magnetic configurations and the plasma parameters are chosen to be almost identical to those in Ref.9, and thus

$$
B/B_0 = 1 - \varepsilon_{\mathrm{t}}(s)\cos\theta_{\mathrm{B}} + \varepsilon_{\mathrm{t}}(s)\left(1 - \cos\theta_{\mathrm{B}}\right)\cos\left(L\theta_{\mathrm{B}} - N\zeta_{\mathrm{B}}\right) \tag{47}
$$

with $L = 1$, $N = 4$, and $0 \leq \varepsilon_{\mathrm{T}} \leq 0.26$ are used for the $\mathbf{B}$-field strength. The $\mathrm{e}^- + \mathrm{D}^+ + \mathrm{C}^{6+}$ multi-ion-species plasma with $Z_{\mathrm{eff}} = 1.9$, $T_{\mathrm{i}} = T_{\mathrm{e}} = 0.5\mathrm{keV}$, $n_{\mathrm{e}} = 1 \times 10^{19}\mathrm{m}^{-3}$ (resultant parameters in Eq.(2) are $v_{\mathrm{c}} = 1.01\mathrm{Mm/s}$ and $Z_2 = 3.70$) that is sustained by a hydrogen beam with $m_{\mathrm{f}} v_{\mathrm{b}}^2/2 = 27\mathrm{keV}$ is assumed. Analogous to the previously investigated parallel momentum input by the tangential NB injections, the dependence on ε_{t} is large when the energy space weighting to low energy regions is large in this kind of integral $\left\langle \int H_2(v) P_2(\xi) f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}/B \right\rangle$.

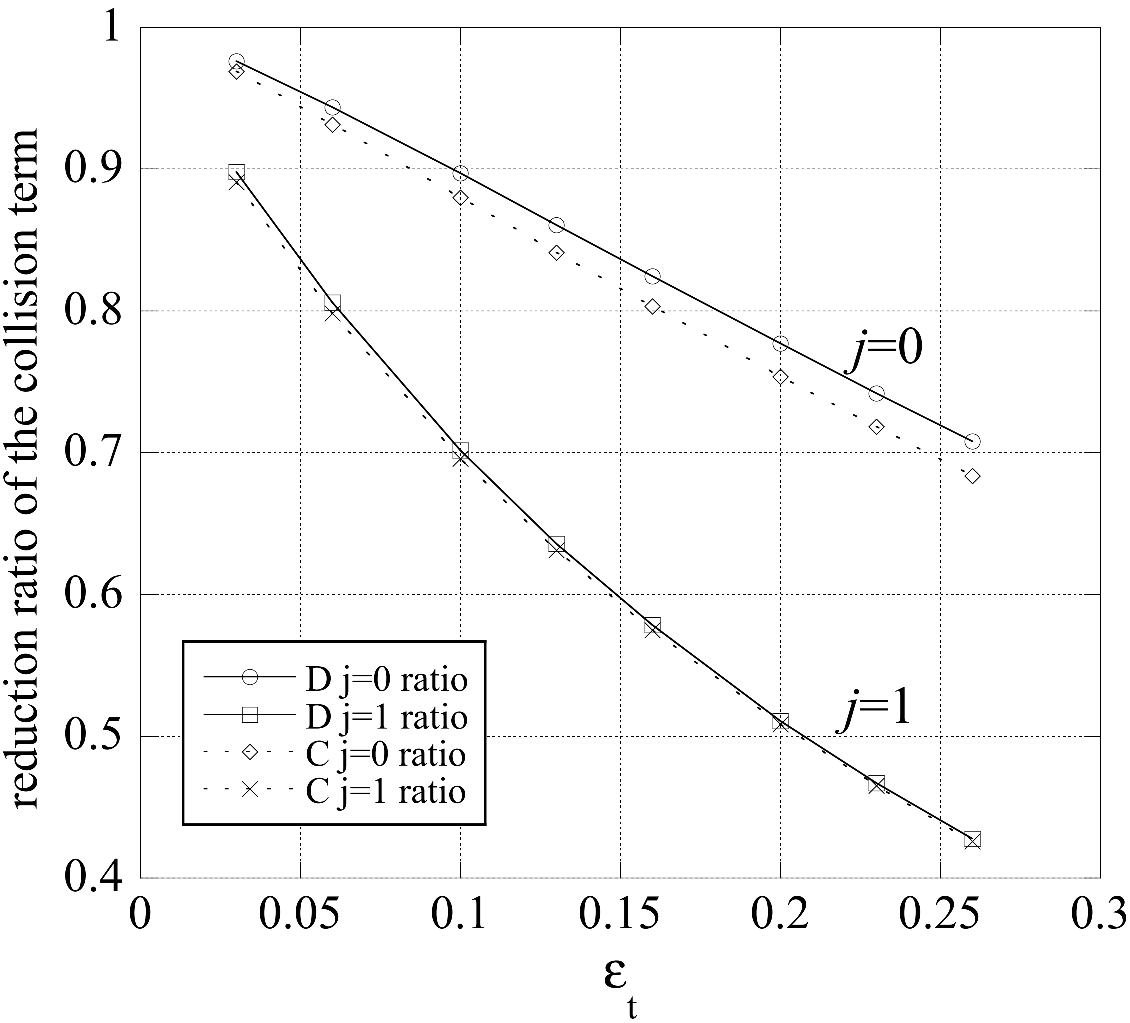

FIG. 2. The configuration dependence of the ion anisotropic heating term $\int x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) C_{a\mathrm{f}}\,(f_{a\mathrm{M}}, f_\mathrm{f})\,\mathrm{d}^3\mathbf{v}$ with $j = 0, 1$ for D^+ and D^{6+} in the model field Eq.(47). The result is normalized by the calculation using Eq.(20) for $\mathbf{b} \cdot \nabla B = 0$.

However, the contrasting fact is that the reduction of these second Legendre order moments due to the $\mathbf{B}$-field strength modulation is proportional to $1 - \langle B \rangle / B_\mathrm{M}$ while the reduction of the first Legendre order moments $\left\langle B \int F(v) \xi f_\mathrm{f} \mathrm{d}^3\mathbf{v} \right\rangle$ was determined by $\left\langle (1 - B/B_\mathrm{M})^{1/2} \right\rangle$. This difference is due to a difference between $\overline{f}_\mathrm{f}^{(\mathrm{odd})}$ that is limited to the pitch-angle range $\lambda \leq 1$ and $\overline{f}_\mathrm{f}^{(\mathrm{even})}$ that is broadened to the full range $0 \leq \lambda \leq B_\mathrm{M}/B$ as mentioned in Secs.I-II.

Next, by using the anisotropic heating term given by this method, we investigate the pressure anisotropy $\left\langle (p_{\|a} - p_{\perp a})/B \right\rangle$ that is given as the response in the LHS of Eq.(42). Based on past experimental and theoretical studies on the Shafranov shifts in the NBI-heated plasmas[7,8,10,11], we shall assume a possibility of $\left| p_{\|\mathrm{f}} - p_{\perp\mathrm{f}} \right| \sim p_{\mathrm{f}} \sim p_{\mathrm{e}} \sim \sum_{a\neq\mathrm{e,f}} p_a$, and investigate the dependence of $\left| \left\langle p_{\|a} - p_{\perp a} \right\rangle \right| / \langle p_a \rangle \sim (\langle p_{\mathrm{e}} \rangle / \langle p_a \rangle) \left| \left\langle (p_{\|a} - p_{\perp a})/B \right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle \right|$ in that condition on the plasma parameters (n_a, T_a) and the beam energy $m_{\mathrm{f}} v_{\mathrm{b}}^2/2$. When considering this problem with a fixed Z_{eff} value, the ratio $\left\langle (p_{\|a} - p_{\perp a})/B \right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle$ is insensitive to the plasma density, and the dependence on it is due to only a weak dependence of the Coulomb logarithm on the density. In addition to this fact, the ratio $\left\langle (p_{\|\mathrm{e}} - p_{\perp\mathrm{e}})/B \right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle$ of the electron anisotropy is insensitive also to the injection velocity v_{b} and the electron temperature $\langle T_{\mathrm{e}} \rangle$ as understood by the $v_{\mathrm{b}}^2 \ll 2T_{\mathrm{e}}/m_{\mathrm{e}}$ approximation in Eq.(46). Furthermore, the electron anisotropy ratio is small because of a mass ratio relation $\left\langle (p_{\|\mathrm{e}} - p_{\perp\mathrm{e}})/B \right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle \sim \tau_{\mathrm{S}}^{-1} / \sum_{a\neq\mathrm{f}} \tau_{\mathrm{e}a}^{-1} \approx m_{\mathrm{e}}/m_{\mathrm{f}}/Z_{\mathrm{eff}}$. In contrast to this simple situation of the electron anisotropy, this kind of simple scaling on the parameters v_{b}, $\langle T_{\mathrm{i}} \rangle \equiv \sum_{a\neq\mathrm{e,f}} \langle p_a \rangle / \sum_{a\neq\mathrm{e,f}} \langle n_a \rangle$, and the masses does not exist for the anisotropic ion heating. Therefore, the anisotropy of the thermalized ions must be investigated by the numerical solution of Eq.(42). The numerical examples of the ratio $(\langle p_{\mathrm{e}} \rangle / \langle p_a \rangle) \left| \left\langle (p_{\|a} - p_{\perp a})/B \right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle \right|$ of the thermalized ions $a \neq$ e,f together with that of the electron are shown in Fig.3. The assumed **B**-field strength is $\varepsilon_{\mathrm{T}} = 0.1$ in Eq.(47). For investigating the temperature dependence, the other parameters are changed from those in Fig.2, and thus we assume the $\mathrm{e}^- + \mathrm{D}^+ + \mathrm{C}^{6+}$ multi-ion-species plasmas with $n_{\mathrm{e}} = 1 \times 10^{19}\mathrm{m}^{-3}$, $Z_{\mathrm{eff}} = 1.9$, and $0.2\mathrm{keV} \leq T_{\mathrm{i}} = T_{\mathrm{e}} \leq 5\mathrm{keV}$ that are sustained by a hydrogen beam with $m_{\mathrm{f}} v_{\mathrm{b}}^2/2 = 50\mathrm{keV}$. The critical velocity $v_{\mathrm{c}} \propto v_{\mathrm{Te}}$ in the fast ion velocity distribution is varying in this temperature scan even though $Z_2 \cong 3.7$ is unchanged. The relation of the injection and the critical velocities becomes $v_{\mathrm{b}} \approx v_{\mathrm{c}}$ at $T_{\mathrm{e}} \approx 5\mathrm{keV}$. This numerical result indicates that ion anisotropy may become large for high-T_{i} conditions (in particular, when $v_{\mathrm{b}} \approx v_{\mathrm{c}}$) while the electrons' anisotropy ratio is insensitive to the temperature following the above scaling based on the $v_{\mathrm{b}}^2 \ll 2T_{\mathrm{e}}/m_{\mathrm{e}}$ approximation of the source term. In these conditions, however, the ions' anisotropy ratios are still $(\langle p_{\mathrm{e}} \rangle / \langle p_a \rangle) \left| \left\langle (p_{\|a} - p_{\perp a})/B \right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle \right| < 10^{-2}$ and thus they can be regarded as isotropic-pressure species for the MHD equilibrium, the DKE Eq.(21), and the classical diffusions. It also should be noted that the ion "anisotropic heating"source term has a polarity

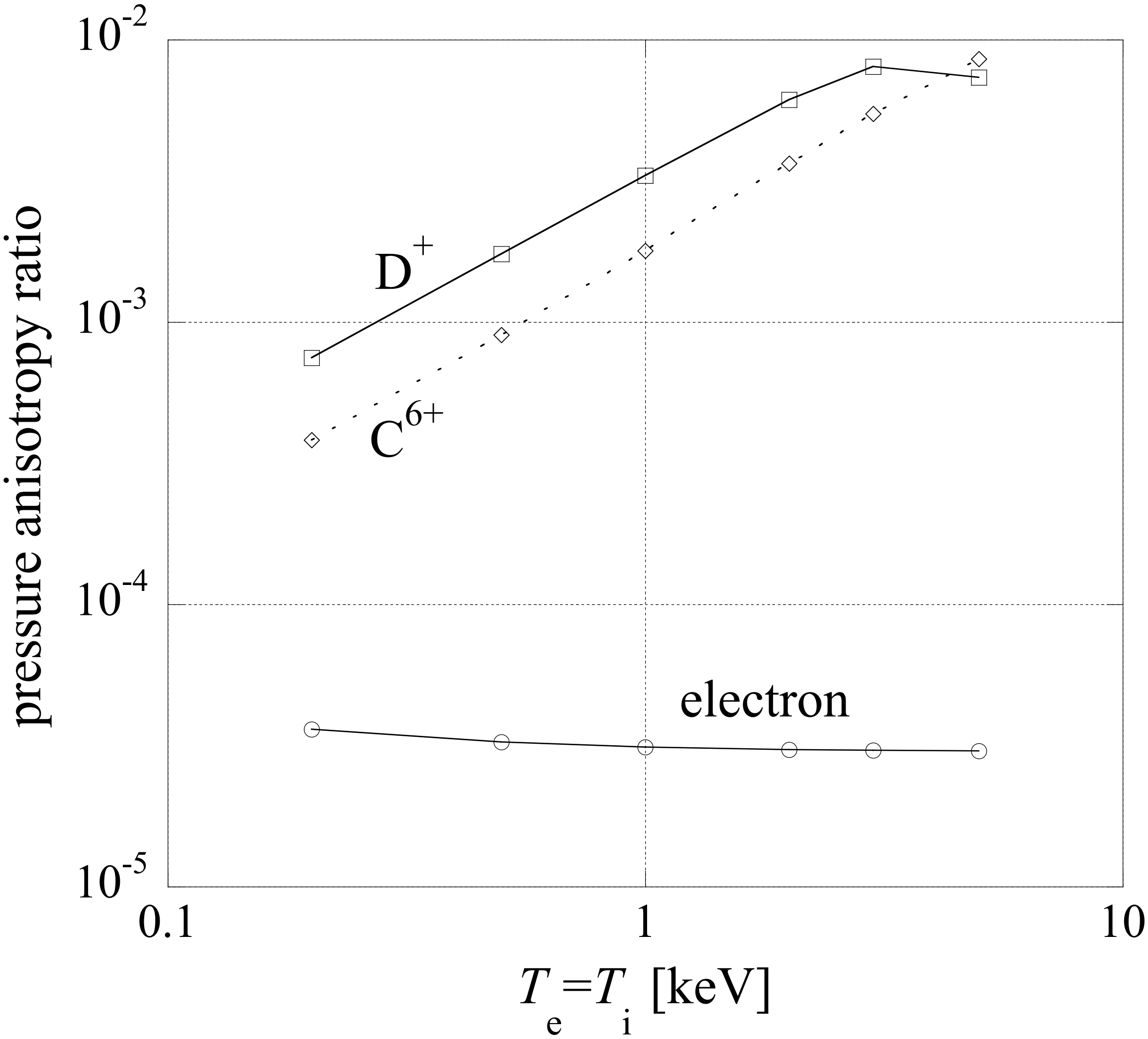

FIG. 3. The temperature dependence of the anisotropy ratios $(\langle p_{\mathrm{e}}\rangle / \langle p_a\rangle) \left|\left\langle (p_{\|a} - p_{\perp a})/B\right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp \mathrm{f}})/B\right\rangle\right|$. The assumed $\mathbf{B}$-field strength is Eq.(47) with $\varepsilon_{\mathrm{t}} = 0.1$, and the beam injection energy is 50keV.

of $\int x_a^2 P_2(\xi) C_{a\mathrm{f}}\left(f_{aM}, f_{\mathrm{f}}\right) \mathrm{d}^3\mathbf{v}/(p_{\|\mathrm{f}} - p_{\perp \mathrm{f}}) < 0$ and the resultant ions' anisotropies also become $\left\langle (p_{\|a} - p_{\perp a})/B\right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp \mathrm{f}})/B\right\rangle < 0$ in these investigated cases. This is a contrasting characteristic compared with the electron heating and the previously investigated beam driven flows. The anisotropy of the thermalized ions tends to cancel that of fast ions in the slow velocity range $v < v_{\mathrm{c}}$.

IV. FLOW AND FRICTION MOMENTS IN PARALLEL AND PERPENDICULAR DIRECTIONS

In this section, we shall develop a method to obtain friction integrals including the slowing down velocity distribution function for the parallel direction for the P-S diffusions $\left\langle \mathbf{\Gamma}_a^{\mathrm{PS}} \cdot \nabla s \right\rangle \equiv -e_a^{-1} c \left\langle \widetilde{U} F_{\parallel\, a1} \right\rangle$, $\left\langle \mathbf{Q}_a^{\mathrm{PS}} \cdot \nabla s \right\rangle \equiv -e_a^{-1} c m_a \left\langle \widetilde{U} G_{\parallel a} \right\rangle$, and the perpendicular direction for the classical diffusions $\left\langle \mathbf{\Gamma}_a^{\mathrm{cl}} \cdot \nabla s \right\rangle \equiv e_a^{-1} c \left\langle B^{-2} \mathbf{F}_{a1} \times \mathbf{B} \cdot \nabla s \right\rangle$, $\left\langle \mathbf{Q}_a^{\mathrm{cl}} \cdot \nabla s \right\rangle \equiv e_a^{-1} c m_a \left\langle B^{-2} \mathbf{G}_a \times \mathbf{B} \cdot \nabla s \right\rangle$ due to $\mathbf{F}_{a1} \equiv m_a \int \mathbf{v} \sum_b C_{ab}\left(f_a, f_b\right) \mathrm{d}^3\mathbf{v}$, $\mathbf{G}_a \equiv (m_a/2) \int \mathbf{v} v^2 \sum_b C_{ab}\left(f_a, f_b\right) \mathrm{d}^3\mathbf{v}$ of both the thermalized target plasma species and the fast ions themselves. The required integrals are $\int \mathbf{v} L_j^{(3/2)}(x_a^2) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right) \mathrm{d}^3\mathbf{v}$ (basically with $j = 0, 1, 2$ for Eq.(28)) of thermalized species and $\int \mathbf{v} v^{2j} \sum_{b \neq \mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v}$ (with $j = 0, 1$) of fast ions. The fast ion friction integrals should be obtained not by Eq.(2) but by the standard formula of the test particle portion $C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right)$. For obtaining the fast ions' gyro-phase-averaged slowing down velocity distribution function $\overline{f}_{\mathrm{f}}\left(\mathbf{x}, v, \sigma, \lambda\right)$ that should be $\left|\overline{f}_{\mathrm{f}}^{(l \geq 1)}\right| \ll \overline{f}_{\mathrm{f}}^{(l=0)}$ (an isotropic structure in the pitch-angle space) and $\left|\partial^2 \overline{f}_{\mathrm{f}}^{(l=0)} / \partial v^2\right| / \overline{f}_{\mathrm{f}}^{(l=0)} \ll m_{\mathrm{f}}/T_{\mathrm{i}}$ (a flat energy space structure) at the thermalized energy range $v^2 \sim 2T_{\mathrm{i}}/m_{\mathrm{f}}$, Eq.(2) includes some minor modifications to the standard formula (mainly, $\left(m_{\mathrm{f}} v_\alpha / T_{\mathrm{e}} + \partial/\partial v_\alpha\right) f_{\mathrm{f}} \cong m_{\mathrm{f}} v_\alpha f_{\mathrm{f}} / T_{\mathrm{e}}$ for the f-e collision and $\left(m_{\mathrm{f}} v / T_{\mathrm{i}} + \partial/\partial v\right) f_{\mathrm{f}} \cong m_{\mathrm{f}} v f_{\mathrm{f}} / T_{\mathrm{i}}$ for the f-i collisions). After determining such $\overline{f}_{\mathrm{f}}\left(\mathbf{x}, v, \sigma, \lambda\right)$ based on Eqs.(1) and (2), various collision integrals such as the momentum/energy transfer to the thermalized target plasma particle species, and $\sum_{b \neq \mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}^{(l=0)}, f_{b\mathrm{M}}\right)$ giving the particle fueling to the thermalized ion species with $m_a = m_{\mathrm{f}}$ and $e_a = e_{\mathrm{f}}$ should be calculated by the standard formulae. A difference between Eq.(2) and the standard formula in the energy transfer $m_{\mathrm{f}} \int v^2 C_{\mathrm{fa}}\left(f_{\mathrm{f}}, f_{a\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} = -m_a \int v^2 C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right) \mathrm{d}^3\mathbf{v}$ and the physical meaning of the difference are already explained in Ref.9, and analogous differences in $\mathbf{G}_{\mathrm{f}} \equiv (m_f/2) \int \mathbf{v} v^2 \sum_{b \neq \mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v}$ and $\int G(x_a) v^{-1} \sum_{b \neq \mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v}$ will be shown in this section. Most of the required friction integral formulas are already shown in Ref.19 except this $\mathbf{G}_{\mathrm{f}}$. It is obtained by using

$$\int v^3 \xi C_{ab}\left(f_a, f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} =$$

$$8\pi^2 n_b \left(\frac{e_a e_b}{m_a}\right)^2 \ln\Lambda_{ab} \int_0^\infty \left\{2\Phi(x_b) + 4G\left(x_b\right) - 3\left(\frac{m_a}{m_b} + 1\right) \frac{m_b v^2}{T_b} G\left(x_b\right)\right\} \left(\int_{-1}^{1} \xi f_a \mathrm{d}\xi\right) v^2 \mathrm{d}v \tag{48}$$

with the error function $\Phi(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\mathrm{d}y$. Firstly, the parallel friction for the P-S diffusions is considered. From the viewpoint of a consistency with the P-S current in the MHD equilibrium, the first order Legendre moment $\int_{-1}^1 \xi\overline{f}_{\mathrm{f}1}\mathrm{d}\xi$ of $\overline{f}_{\mathrm{f}1}$ as the first order of ρ_{f}/L_r basically has a form of[9]

$$\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}1}\mathrm{d}\xi-\left\langle B\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}1}\mathrm{d}\xi\right\rangle\frac{B}{\langle B^2\rangle}=-\frac{c}{e_{\mathrm{f}}}\frac{m_{\mathrm{f}}v}{4}\widetilde{U}\frac{\partial}{\partial s}\left\langle\int_{-1}^{1}\left(1+\xi^2\right)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle. \tag{49}$$

For general particle species a in general toroidal plasmas, the response to $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial\overline{f}_a^{(\mathrm{even})}/\partial s$ usually includes the spontaneously generated integration constant $\left\langle B\int_{-1}^1\xi\overline{f}_{a1}\mathrm{d}\xi\right\rangle$ of the particle/energy balance equation (corresponding to the well-know bootstrap current). This generation also should be investigated for fusion-born fast ions,[24] and thus a calculation method for non-symmetric stellarator/heliotron plasmas will be reported in a separated article. For the NB-produced fast ions in the tangential NBI operations, however, the 0th order of ρ_{f}/L_r determined by Eq.(1) already includes the large first Legendre order component $\left\langle B\int_{-1}^1\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle$ as the response to the fast ion source term that is already investigated for non-symmetric stellarator/heliotron configurations in Ref.9. Therefore the integration constant in the first order of ρ_{f}/L_r is neglected here. Basically, the parallel friction integrals must be obtained by substituting Eq.(49) into the aforementioned formulas of $\int v\xi L_j^{(3/2)}(x_a^2)C_{a\mathrm{f}}\left(f_{a\mathrm{M}},f_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v}$ and $\int v^{2j+1}\xi\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(f_{\mathrm{f}},f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}$. For the lowest Legendre order component $\frac{\partial}{\partial s}\left\langle\int_{-1}^1\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle$ in the RHS in Eq.(49), an analytical solution that can be commonly used for this purpose in general toroidal configurations is already known.[9] The application of the adjoint equation method in Sec.II for obtaining $\left\langle\int v^k f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\right\rangle$ with $k\geq-2$ is easy and straightforward, and supports the validity of the analytical expression. However, the energy space structure of $\frac{\partial}{\partial s}\left\langle\int_{-1}^1 P_2(\xi)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle$ cannot be given by this method since it is a method to obtain $\left\langle\int\mathrm{d}^3\mathbf{v}\right\rangle$ integrals in Eq.(5). Instead of $\frac{\partial}{\partial s}\left\langle\int_{-1}^1 P_2(\xi)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle$, we shall use

$$\int v^{k-1}\xi\overline{f}_{\mathrm{f}1}\mathrm{d}^3\mathbf{v}-\left\langle B\int v^{k-1}\xi\overline{f}_{\mathrm{f}1}\mathrm{d}^3\mathbf{v}\right\rangle\frac{B}{\langle B^2\rangle}=-\frac{c}{e_{\mathrm{f}}}\frac{m_{\mathrm{f}}}{4}\widetilde{U}\frac{\partial}{\partial s}\left\langle\int v^k\left(1+\xi^2\right)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\right\rangle \tag{50}$$

with $k=-1,1,2,4,6$ as $\int_0^\infty\mathrm{d}v$ integrals of Eq.(49), and use Eqs.(5) and (16) for obtaining $\left\langle\int v^k P_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\right\rangle/\langle B^{-1}\rangle$ in this RHS. The numerical differential of the radial distribution of the numerically obtained $\left\langle\int v^k P_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\right\rangle/\langle B^{-1}\rangle$ will be adequate for this purpose since the poloidal/toroidal variations of the NB-produced fast ions' anisotropy in the tangential NBI operations are $\int v^k P_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\propto B(\theta,\zeta)^{\pm1}$ at most. As shown below, the required friction integrals are obtained by using only these radial gradients.

The approximated friction formulas for thermalized ions use an approximation $G(x_b) \cong (2x_b^2)^{-1}$ for the situation $\left|\partial f_{\mathrm{f}}^{(l=2)}/\partial s\right| \ll \left|\partial f_{\mathrm{f}}^{(l=0)}/\partial s\right|$ at $v^2 \sim 2T_{\mathrm{i}}/m_{\mathrm{f}}$ as follows:

$$\begin{aligned} m_a \int v\xi C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right) \mathrm{d}^3\mathbf{v} &= -m_{\mathrm{f}} \int v\xi C_{\mathrm{f}a}\left(f_{\mathrm{f}}, f_{a\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} \\ &= 8\pi^2 \frac{n_a\left(e_a e_{\mathrm{f}}\right)^2 \ln\Lambda_{a\mathrm{f}}}{m_a}\left(\frac{m_a}{m_{\mathrm{f}}}+1\right)\int_0^{v_{\mathrm{b}}}\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)\mathrm{d}v, \\ m_a \int v\xi L_1^{(3/2)}\left(x_a^2\right) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v} &= -24\pi^2\frac{n_a\left(e_a e_{\mathrm{f}}\right)^2\ln\Lambda_{a\mathrm{f}}}{m_a}\int_0^{v_{\mathrm{b}}}\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)\mathrm{d}v, \\ m_a \int v\xi L_j^{(3/2)}\left(x_a^2\right) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v} &= 0 \text{ for } j \geq 2. \end{aligned} \tag{51}$$

These are obtained by using only $\int v^{-2}\xi\overline{f}_{\mathrm{f}1}\mathrm{d}^3\mathbf{v}$ driven by $\frac{\partial}{\partial s}\left(\left\langle \int v^{-1}P_2(\xi) f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\right\rangle / \left\langle B^{-1}\right\rangle\right)$. For the approximated friction formulas for electrons, we shall basically use the first three terms in the power series expressions

$$G(x) = \frac{2}{\sqrt{\pi}} x \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+3)n!}x^{2n}, \;\; \exp\left(-x^2\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}.$$

Therefore, the e-f friction integrals are given by

$$\begin{aligned} m_{\mathrm{e}} \int v\xi C_{\mathrm{ef}}\left(f_{\mathrm{eM}} f_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v} &= -m_{\mathrm{f}}\int v\xi C_{\mathrm{fe}}\left(f_{\mathrm{f}}, f_{\mathrm{eM}}\right)\mathrm{d}^3\mathbf{v} \\ = 16\pi^{3/2}\frac{n_{\mathrm{e}}\left(e e_{\mathrm{f}}\right)^2\ln\Lambda_{\mathrm{ef}}}{T_{\mathrm{e}}} &\int_0^{v_{\mathrm{b}}} x_{\mathrm{e}}\left(\frac{1}{3}-\frac{x_{\mathrm{e}}^2}{5}+\frac{x_{\mathrm{e}}^4}{14}\right)\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)v^2\mathrm{d}v, \\ m_{\mathrm{e}}\int v\xi L_1^{(3/2)}\left(x_{\mathrm{e}}^2\right) C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v} &= \\ 24\pi^{3/2}\frac{n_{\mathrm{e}}\left(e e_{\mathrm{f}}\right)^2\ln\Lambda_{\mathrm{ef}}}{T_{\mathrm{e}}} &\int_0^{v_{\mathrm{b}}} x_{\mathrm{e}}\left(\frac{1}{3}-\frac{3}{5}x_{\mathrm{e}}^2+\frac{5}{14}x_{\mathrm{e}}^4\right)\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)v^2\mathrm{d}v, \\ m_{\mathrm{e}}\int v\xi L_2^{(3/2)}\left(x_{\mathrm{e}}^2\right) C_{\mathrm{ef}}\left(f_{\mathrm{eM}}, f_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v} &= \\ 10\pi^{3/2}\frac{n_{\mathrm{e}}\left(e e_{\mathrm{f}}\right)^2\ln\Lambda_{\mathrm{ef}}}{T_{\mathrm{e}}} &\int_0^{v_{\mathrm{b}}} x_{\mathrm{e}}\left(1-3x_{\mathrm{e}}^2+\frac{5}{2}x_{\mathrm{e}}^4\right)\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)v^2\mathrm{d}v. \end{aligned} \tag{52}$$

When $v_{\mathrm{b}}/v_{\mathrm{Te}} < 0.8$, the e-f friction integrals can be obtained by these formulas that require only $\int v^{k-1}\xi\overline{f}_{\mathrm{f}1}\mathrm{d}^3\mathbf{v}$ driven by $\frac{\partial}{\partial s}\left(\left\langle\int v^k P_2(\xi) f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\right\rangle / \left\langle B^{-1}\right\rangle\right)$ with $k = 2, 4, 6$. When more accurate calculations for a wide range of $v_{\mathrm{b}}/v_{\mathrm{Te}}$ ratio are required, it is better to obtain these integrals having a common form $\int_0^{v_{\mathrm{b}}} F\left(x_{\mathrm{e}}\right)\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)v^3\mathrm{d}v$, in which $F\left(x_{\mathrm{e}}\right)$ is a moderately varying function and $\int_{-1}^{1} \xi \overline{f}_{\mathrm{f}} \mathrm{d}\xi \propto v^{-2}$ at $v > v_{\mathrm{c}}$, by a polynomial expression

$$\begin{aligned} & F\left(x_{\mathrm{e}} \leq \frac{v_{\mathrm{b}}}{v_{\mathrm{Te}}}\right) \\ & \cong \frac{3}{2} F\left(\frac{v_{\mathrm{b}}}{3 v_{\mathrm{Te}}}\right)-\frac{3}{5} F\left(\frac{2 v_{\mathrm{b}}}{3 v_{\mathrm{Te}}}\right)+\frac{1}{10} F\left(\frac{v_{\mathrm{b}}}{v_{\mathrm{Te}}}\right) \\ & -3\left\{\frac{13}{8} F\left(\frac{v_{\mathrm{b}}}{3 v_{\mathrm{Te}}}\right)-2 F\left(\frac{2 v_{\mathrm{b}}}{3 v_{\mathrm{Te}}}\right)+\frac{3}{8} F\left(\frac{v_{\mathrm{b}}}{v_{\mathrm{Te}}}\right)\right\}\left(\frac{v}{v_{\mathrm{b}}}\right)^{2} \\ & +\frac{27}{5}\left\{\frac{5}{8} F\left(\frac{v_{\mathrm{b}}}{3 v_{\mathrm{Te}}}\right)-F\left(\frac{2 v_{\mathrm{b}}}{3 v_{\mathrm{Te}}}\right)+\frac{3}{8} F\left(\frac{v_{\mathrm{b}}}{v_{\mathrm{Te}}}\right)\right\}\left(\frac{v}{v_{\mathrm{b}}}\right)^{4} . \end{aligned} \tag{53}$$

This formula converges to the first three terms in the power series expression of $F\left(x_{\mathrm{e}}\right)$ when $\left(v_{\mathrm{b}}/v_{\mathrm{Te}}\right)^{2} \ll 1$. (The purpose for which we consider a calculation method of $\int v^{k-1} \xi \overline{f}_{\mathrm{f1}} \mathrm{d}^{3}\mathbf{v}$ with $k=4,6$ below is not in these integrals themselves with large k values but in the e-f friction integrals where the weighting of the high energy range $v \sim v_{\mathrm{b}}$ is reduced.) A method for $\mathbf{F}_{\mathrm{f1}} \equiv m_{\mathrm{f}} \int \mathbf{v} \sum_{b \neq \mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right) \mathrm{d}^{3}\mathbf{v}$ of the fast ions themselves is already obvious because of the momentum conserving relations in Eqs.(51) and (52). The formula for the energy weighted friction integral $\mathbf{G}_{\mathrm{f}}$ is obtained by summing Eq.(48), and using $\Phi\left(x_{b}\right) \cong 1$, $G\left(x_{b}\right) \cong \left(2 x_{b}^{2}\right)^{-1}$ for the f-i collision $b \neq$ e,f as follows:

$$\begin{aligned} & \int v^{3} \xi \sum_{b \neq \mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right) \mathrm{d}^{3}\mathbf{v} \\ & =-\frac{2 \pi}{\tau_{\mathrm{S}}} \int_{0}^{v_{\mathrm{b}}}\left\{3 v^{2} v_{\mathrm{Te}} \frac{3 \sqrt{\pi}}{2} G\left(x_{\mathrm{e}} \leq \frac{v_{\mathrm{b}}}{v_{\mathrm{Te}}}\right)+v_{\mathrm{c}}^{3}\left(3+Z_{2}-\frac{4 T_{\mathrm{i}}}{m_{\mathrm{f}} v^{2}}\right)\right\}\left(\int_{-1}^{1} \xi \overline{f}_{\mathrm{f}} \mathrm{d}\xi\right) v^{2} \mathrm{d}v . \end{aligned} \tag{54}$$

This calculation also requires only $\frac{\partial}{\partial s}\left(\left\langle\int v^{k} P_{2}(\xi) f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}/B\right\rangle / \left\langle B^{-1}\right\rangle\right)$ with $k=-1,1,4,6$. For covering a wide range of the $v_{\mathrm{b}}/v_{\mathrm{Te}}$ ratio only by these limited integrals, the Chandrasekhar function $G\left(x_{\mathrm{e}}\right)$ is calculated by a polynomial expression

$$G\left(x_{\mathrm{e}} \leq \frac{v_{\mathrm{b}}}{v_{\mathrm{Te}}}\right) \cong \frac{v}{v_{\mathrm{b}}}\left[2 \sqrt{2} G\left(\frac{v_{\mathrm{b}}}{\sqrt{2} v_{\mathrm{Te}}}\right)-G\left(\frac{v_{\mathrm{b}}}{v_{\mathrm{Te}}}\right)-2\left\{\sqrt{2} G\left(\frac{v_{\mathrm{b}}}{\sqrt{2} v_{\mathrm{Te}}}\right)-G\left(\frac{v_{\mathrm{b}}}{v_{\mathrm{Te}}}\right)\right\}\left(\frac{v}{v_{\mathrm{b}}}\right)^{2}\right] . \tag{55}$$

Although $\Phi\left(x_{\mathrm{e}}\right)+2 G\left(x_{\mathrm{e}}\right)$ can be retained in the f-e collision integral itself, this term is negligible in this total $\int v^{3} \xi \sum_{b \neq \mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right) \mathrm{d}^{3}\mathbf{v}$ integral formula. Therefore, an essential difference between the $\int \mathbf{v} v^{2} \mathrm{d}^{3}\mathbf{v}$ integrals of Eq.(2) and the standard formula of $\sum_{b \neq \mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right)$ exists only in the last term $\propto 4 T_{\mathrm{i}} / \left(m_{\mathrm{f}} v^{2}\right)$ in Eq.(54) originated in collisions with thermalized ions $b \neq$ e,f. In the straightforward use of the standard formula instead of Eq.(2), f_{f} at $m_{\mathrm{f}} v^{2} \sim 4 T_{\mathrm{i}} / \left(3+Z_{2}\right)$ cannot contribute effectively to $\left\langle\mathbf{Q}_{\mathrm{f}}^{\mathrm{PS}} \cdot \nabla s\right\rangle$ and $\left\langle\mathbf{Q}_{\mathrm{f}}^{\mathrm{cl}} \cdot \nabla s\right\rangle$. The radial transport of this thermalized energy range must be handled in the kinetic equation for the thermalized ion species with $m_a = m_\mathrm{f}$ and $e_a = e_\mathrm{f}$, and the $f_\mathrm{f}\left(v^2 \sim 2T_\mathrm{i}/m_\mathrm{f}\right)$ has only a role as a particle source to that species. Therefore, Eq.(2) overestimating the contribution of $m_\mathrm{f}v^2 \sim 4T_\mathrm{i}/\left(3+Z_2\right)$ is not used for the purpose of the P-S and the classical energy diffusions.

On the parallel flow moments given by Eq.(50) and the resultant parallel friction integrals, it also should be noted that, when the first order of $(v\tau_\mathrm{S})^{-1}$ that will appear in the straightforward solving procedure of Eq.(1) (i.e., the procedure of Cordey mentioned in the introduction) is included, the real space structure of parallel flow moments will deviate from the form $\propto \widetilde{U}$. Although the parallel particle flux given by $k = 2$ of Eq.(51) has this form for the consistency with the $\mathbf{J}$-vector field in the MHD equilibrium, other parallel flow moments $k \neq 2$ can deviate from this form without any inconsistencies. The existence of $\left\langle \mathbf{b}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_\mathrm{f}\mathrm{d}\xi/B\right)\right\rangle \neq 0$ that consists of $\int_{-1}^{1}\xi\overline{f}_\mathrm{f}\mathrm{d}\xi$ being $\propto (v\tau_\mathrm{S})^{-1}$ in Eq.(18) and with the anti-symmetric phase $F\left(-\theta,-\zeta\right) = -F\left(\theta,\zeta\right)$ is obvious in comparisons of Eq.(20) neglecting this term and Eq.(5) for including it. Examples for $\left\langle B^{-1}\int x_a^2 L_j^{(5/2)}(x_a^2)P_2(\xi)C_{a\mathrm{f}}\left(f_{a\mathrm{M}},\ f_\mathrm{f}\right)\mathrm{d}^3\mathbf{v}\right\rangle$ with $j = 0, 1$ in the DKE for thermalized ions were already shown in Sec.III. In the comparisons for $\left\langle\int v^k P_2(\xi)f_\mathrm{f}\mathrm{d}^3\mathbf{v}/B\right\rangle$ that will be shown below, the contribution of the $\left\langle \mathbf{b}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_\mathrm{f}\mathrm{d}\xi/B\right)\right\rangle \neq 0$ in the DKE term balance is large for $k \sim -1$ and small for $k \geq 2$. These results suggest that the $\int_{-1}^{1}\xi\overline{f}_\mathrm{f}\mathrm{d}\xi$ with the anti-symmetric phase substantially exists only in a relatively low energy range $v \lesssim v_\mathrm{c}$ analogous to the $\left\langle\left(\int_{-1}^{1}P_2(\xi)\overline{f}_\mathrm{f}\mathrm{d}\xi\right)\mathbf{B}\cdot\nabla\mathrm{ln}B\right\rangle$ as the parallel viscosity force of the fast ions themselves.[9] Therefore, $k \geq 2$ of Eq.(50) determined by the full energy range $0 \leq v \leq v_\mathrm{b}$ is basically correct even when the first order of $(v\tau_\mathrm{S})^{-1}$ is included. The deviation may appear in integrals that are determined only by the limited energy range $v \lesssim v_\mathrm{c}$ such as the momentum exchange between the fast and the thermalized ions in Eq.(51). Even in this low energy range, the $\int_{-1}^{1}\xi\overline{f}_\mathrm{f}\mathrm{d}\xi$ in Eq.(1) is estimated to be small by the following comparison of $\int_{-1}^{1}\mathrm{d}\xi$ integrals of Eq.(1) and $\left(V_\parallel - C_\mathrm{f}\right)\overline{f}_{\mathrm{f}1} = -\mathbf{v}_\mathrm{df}\cdot\nabla\overline{f}_\mathrm{f}^{(\mathrm{even})}$. The integral of Eq.(1) is

$$\begin{aligned}\mathbf{B}\cdot\nabla\frac{\int_{-1}^{1}\xi\overline{f}_\mathrm{f}\mathrm{d}\xi}{B} &= v^{-1}\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(\int_{-1}^{1}\overline{f}_\mathrm{f}\mathrm{d}\xi - \left\langle\int_{-1}^{1}\overline{f}_\mathrm{f}\mathrm{d}\xi\right\rangle, f_{b\mathrm{M}}\right)\\ &+ v^{-1}\left(\int_{-1}^{1}S_\mathrm{f}(s,v,\sigma,\lambda)\mathrm{d}\xi - \left\langle\int_{-1}^{1}S_\mathrm{f}(s,v,\sigma,\lambda)\mathrm{d}\xi\right\rangle\right)\end{aligned} \tag{56}$$

and is compared with

$$\mathbf{B}\cdot\nabla\frac{\int_{-1}^{1}\xi\overline{f}_{\mathrm{f1}}\mathrm{d}\xi}{B}=\frac{c}{4e_{\mathrm{f}}}m_{\mathrm{f}}v\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^{2}}\frac{\partial}{\partial s}\left\langle\int_{-1}^{1}\left(1+\xi^{2}\right)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle$$
$$=\frac{c}{4e_{\mathrm{f}}}m_{\mathrm{f}}v\frac{B^{2}}{\left\langle B^{2}\right\rangle^{2}}\frac{4\pi^{2}}{V'}\sum_{(m,n)\neq(0,0)}\left(B_{\zeta}^{(\mathrm{Boozer})}m+B_{\theta}^{(\mathrm{Boozer})}n\right)\varepsilon_{mn}^{(\mathrm{Boozer})}\sin(m\theta_{\mathrm{B}}-n\zeta_{\mathrm{B}}) \quad (57)$$
$$\times\frac{\partial}{\partial s}\left\langle\int_{-1}^{1}\left(1+\xi^{2}\right)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle$$

that determines the response to $\mathbf{v}_{\mathrm{df}}\cdot\nabla\overline{f}_{\mathrm{f}}^{(\mathrm{even})}$. Hereafter, the covariant expression of the $\mathbf{B}$-vector field

$$\mathbf{B}=B_{s}^{(\mathrm{Boozer})}\nabla s+B_{\theta}^{(\mathrm{Boozer})}\nabla\theta_{\mathrm{B}}+B_{\zeta}^{(\mathrm{Boozer})}\nabla\zeta_{\mathrm{B}} \quad (58)$$

in the Boozer coordinates is assumed. In Eq.(56), we should exclude the surface-averaged term $\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(\left\langle\int_{-1}^{1}\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle,f_{b\mathrm{M}}\right)+\left\langle\int_{-1}^{1}S_{\mathrm{f}}(s,v,\sigma,\lambda)\mathrm{d}\xi\right\rangle$ that includes the particle source to the thermalized ion species with $m_a=m_{\mathrm{f}}$ and $e_a=e_{\mathrm{f}}$. This averaged term balances not with the operator $V_{\|}\overline{f}_{\mathrm{f}}$ but with $\frac{\partial}{\partial t}n_a+\frac{\partial}{\partial V}\left\langle n_a\mathbf{u}_a\cdot\nabla V\right\rangle$ of that ion species at the thermalized energy range $v^2\sim 2T_{\mathrm{i}}/m_{\mathrm{f}}$. We will compare RMS(root mean square)s $\sqrt{\left\langle\left(\mathbf{B}\cdot\nabla\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi/B\right)^{2}\right\rangle}$ and $\sqrt{\left\langle\left(\mathbf{B}\cdot\nabla\int_{-1}^{1}\xi\overline{f}_{\mathrm{f1}}\mathrm{d}\xi/B\right)^{2}\right\rangle}$ after following investigation on the energy space structure of the collision term $\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(\int_{-1}^{1}\overline{f}_{\mathrm{f}}\mathrm{d}\xi,f_{b\mathrm{M}}\right)$, because these are oscillating functions in the (θ,ζ) space with the symmetric and the anti-symmetric phases $F\left(-\theta,-\zeta\right)=F\left(\theta,\zeta\right)$, $F\left(-\theta,-\zeta\right)=-F\left(\theta,\zeta\right)$, respectively. Since the purpose of this comparison is in the momentum exchange integrals being $\propto\int_0^\infty v^2G\left(x_a\right)\left(\int_{-1}^{1}\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right)\mathrm{d}v$, we shall derive an integral formula of $\int_0^\infty vG(x_a)\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(\overline{f}_{\mathrm{f}}^{(l=0)},f_{b\mathrm{M}}\right)\mathrm{d}v$ of individual thermalized target particle species a. Integrations by parts using a boundary condition

$$\left[G\left(x_b\right)\left\{\frac{1}{v}\left(3G\left(x_a\right)-\frac{2}{\sqrt{\pi}}x_a\exp\left(-x_a^2\right)\right)+G\left(x_a\right)\left(\frac{m_{\mathrm{f}}v}{T_b}+\frac{\partial}{\partial v}\right)\right\}f_{\mathrm{f}}\right]_{v=0,\infty}=0$$

give

$$
\begin{aligned}
&4\pi\left(\frac{e_{\mathrm{f}}e}{m_{\mathrm{f}}}\right)^2\int_0^\infty G\left(x_a\right)v^{-1}\frac{\partial}{\partial v}\left\{v\sum_{b\neq\mathrm{f}}n_bZ_b^2\ln\Lambda_{\mathrm{f}b}G\left(x_b\right)\left(\frac{m_{\mathrm{f}}v}{T_b}+\frac{\partial}{\partial v}\right)f_{\mathrm{f}}\right\}\mathrm{d}v\\
&=4\pi\left(\frac{e_{\mathrm{f}}e}{m_{\mathrm{f}}}\right)^2\int_0^\infty\frac{1}{v^2}\left[\left(3G\left(x_a\right)-\frac{2}{\sqrt{\pi}}x_a\exp\left(-x_a^2\right)\right)\right.\\
&\qquad\times\sum_{b\neq\mathrm{f}}n_bZ_b^2\ln\Lambda_{\mathrm{f}b}\left\{2\left(1+\frac{m_{\mathrm{f}}}{m_b}x_b^2\right)G\left(x_b\right)+3G\left(x_b\right)-\frac{2}{\sqrt{\pi}}x_b\exp\left(-x_b^2\right)\right\}\\
&\qquad\qquad\left.-\frac{4}{\sqrt{\pi}}x_a^3\exp\left(-x_a^2\right)\sum_{b\neq\mathrm{f}}n_bZ_b^2\ln\Lambda_{\mathrm{f}b}G\left(x_b\right)\right]f_{\mathrm{f}}\mathrm{d}v\\
&\simeq\frac{1}{\tau_{\mathrm{S}}}\int_0^\infty\frac{1}{v^2}\left[\left(3G\left(x_a\right)-\frac{2}{\sqrt{\pi}}x_a\exp\left(-x_a^2\right)\right)\left(v^2v_{\mathrm{Te}}\frac{3\sqrt{\pi}}{2}G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3\right)\right.\\
&\quad\left.-\frac{2}{\sqrt{\pi}}\frac{m_a}{m_{\mathrm{f}}}\frac{T_{\mathrm{e}}}{T_a}x_a\exp\left(-x_a^2\right)\left(v^2v_{\mathrm{Te}}\frac{3\sqrt{\pi}}{2}G\left(x_{\mathrm{e}}\right)+\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}v_{\mathrm{c}}^3\right)\right]f_{\mathrm{f}}\mathrm{d}v\ \text{if}\ f_{\mathrm{f}}\left(v^2\sim 2T_{\mathrm{i}}/m_{\mathrm{f}}\right)=0
\end{aligned}
\tag{59}
$$

This calculation also includes the function $\int_0^x y^4\exp\left(-y^2\right)\mathrm{d}y$ that is already explained for the $\int x_a^2L_j^{(5/2)}\left(x_a^2\right)P_2(\xi)C_{ab}\left(f_{a\mathrm{M}},f_b\right)\mathrm{d}^3\mathbf{v}$ integrals in Sec.III. In the second equality for cases of $f_{\mathrm{f}}\left(v^2\sim 2T_{\mathrm{i}}/m_{\mathrm{f}}\right)=0$, the first term including the $\int_0^{x_a}y^4\exp\left(-y^2\right)\mathrm{d}y$ that gives a positive contribution can be obtained also by the $\int_0^\infty vG(x_a)\left(C_{\mathrm{f}}f_{\mathrm{f}}^{(l=0)}\right)\mathrm{d}v$ integral of Eq.(2). Analogously to the energy transfer integral and the energy weighted friction integral $\int\mathbf{v}v^2\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(f_{\mathrm{f}},f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}$, an essential difference between Eq.(2) and $\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(f_{\mathrm{f}},f_{b\mathrm{M}}\right)$ exists in the second term $\propto\exp\left(-x_a^2\right)$ making a negative contribution. The existence of the broadened tail component of $\mathbf{B}\cdot\nabla\left(\int_{-1}^1\xi\overline{f}_{\mathrm{f}}\mathrm{d}\xi/B\right)$ at $v^2>2T_{\mathrm{i}}/m_{\mathrm{f}}$, which must be handled by the fast ions' DKE Eq.(1) with the collision term Eq.(2), will be indicated by the positive contribution of the first term. On the other hand, the second term $\propto\exp\left(-x_a^2\right)$ in the integrals for the thermalized ions $a\neq$ e,f is only a contribution of the $f_{\mathrm{f}}\left(v^2\sim 2T_{\mathrm{i}}/m_a\right)$ at the thermalized energy range. When this integral formula is used for the purpose of Eq.(56), this contribution corresponds to the possibility of small poloidal and toroidal variations of the particle fueling effect that is order of the aforementioned $\frac{\partial}{\partial t}n_a+\frac{\partial}{\partial V}\left\langle n_a\mathbf{u}_a\cdot\nabla V\right\rangle$. We shall exclude this term $\propto\exp\left(-x_a^2\right)$ with $a\neq$ e,f. Eq.(5) of the adjoint equation method is useful also for investigating this collision integral. Analogous to Eq.(18), the $\left\langle B^{-1}\int_0^\infty v^2G(x_a)\mathrm{d}v\right\rangle$ integral of Eq.(56) is used. The boundary condition at $v=0$ required for this calculation is automatically satisfied by straightforward calculations of the functions $G(x)$ and $3G\left(x\right)-\frac{2}{\sqrt{\pi}}x\exp\left(-x^2\right)$ without any approximations. To solve this adjoint equation

$\left(V_\parallel + C_\mathrm{f}^\mathrm{A}\right) f_\mathrm{A0} = H_0(v)\left(1/B - \langle 1/B\rangle\right)/\tau_\mathrm{S}$ for the lowest Legendre order $l = 0$ is easy and straightforward, and the solution is

$$f_\mathrm{A0}(v = v_\mathrm{b},\ \lambda < 1) = -\frac{1}{B_\mathrm{M}}\int_0^{v_\mathrm{b}} \frac{v^2 H_0(v)}{v^2 v_\mathrm{Te}(3\sqrt{\pi}/2)G\left(x_\mathrm{e}\right) + v_\mathrm{c}^3}$$
$$\times\left[1 - \left\langle\frac{B_\mathrm{M}}{B}\right\rangle + \sum_n \frac{\left\langle (B_\mathrm{M}/B - 1)\int_0^1 \Lambda_n \left\{\partial\left(1 - \lambda B/B_\mathrm{M}\right)^{1/2}/\partial\lambda\right\}\mathrm{d}\lambda\right\rangle}{\left\langle \int_0^1 \Lambda_n^2 \left\{\partial\left(1 - \lambda B/B_\mathrm{M}\right)^{1/2}/\partial\lambda\right\}\mathrm{d}\lambda\right\rangle}\Lambda_n(\lambda)\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_\mathrm{b})}\right\}^{\kappa_n Z_2/3}\right]\mathrm{d}v. \tag{60}$$

This solution is used for the first term in

$$B_\mathrm{M}\left\langle \mathbf{b}\cdot\nabla\frac{\int \xi G\left(x_a\right)\overline{f}_\mathrm{f}\mathrm{d}^3\mathbf{v}}{B}\right\rangle$$
$$= \left\langle \frac{B_\mathrm{M}}{B}\int G\left(x_a\right)v^{-1}\sum_{b\neq\mathrm{f}} C_\mathrm{fb}\left(\overline{f}_\mathrm{f} - \left\langle \overline{f}_\mathrm{f}^{(l=0)}\right\rangle, f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}\right\rangle \tag{61}$$
$$+ 2\pi S_0 G\left(\frac{v_\mathrm{b}}{v_{\mathrm{T}a}}\right)v_\mathrm{b}^{-1}\left(\frac{B_\mathrm{M}\left\langle (1 - \lambda_\mathrm{b}B/B_\mathrm{M})^{-1/2}\right\rangle}{\left\langle B(1 - \lambda_\mathrm{b}B/B_\mathrm{M})^{-1/2}\right\rangle} - \left\langle\frac{B_\mathrm{M}}{B}\right\rangle\right).$$

A relation

$$1 - \left\langle\frac{B_\mathrm{M}}{B}\right\rangle + \sum_n \frac{\left\langle (B_\mathrm{M}/B - 1)\int_0^1 \Lambda_n \left\{\partial\left(1 - \lambda B/B_\mathrm{M}\right)^{1/2}/\partial\lambda\right\}\mathrm{d}\lambda\right\rangle}{\left\langle \int_0^1 \Lambda_n^2 \left\{\partial\left(1 - \lambda B/B_\mathrm{M}\right)^{1/2}/\partial\lambda\right\}\mathrm{d}\lambda\right\rangle}\Lambda_n(\lambda_\mathrm{b})\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_\mathrm{b})}\right\}^{\kappa_n Z_2/3}$$
$$\simeq \begin{cases} 1 - \left\langle\dfrac{B_\mathrm{M}}{B}\right\rangle \text{ at } v^3 \ll v_\mathrm{c}^3 \\ 1 - \left\langle\dfrac{B_\mathrm{M}}{B}\right\rangle + \left[\left\langle\dfrac{B}{B_\mathrm{M}}\dfrac{v}{v_\parallel}\right\rangle^{-1}\left\langle\dfrac{v}{v_\parallel}\left(1 - \dfrac{B}{B_\mathrm{M}}\right)\right\rangle\right]_{\lambda=\lambda_\mathrm{b}} \approx 0 \text{ at } v^3 \gg v_\mathrm{c}^3 \end{cases}$$

indicates that the first Legendre order of the velocity distribution $\int_{-1}^1 \xi\overline{f}_\mathrm{f}\mathrm{d}\xi$ and the lowest Legendre order of the collision term $\int_{-1}^1 C_\mathrm{fb}\left(\overline{f}_\mathrm{f}, f_{b\mathrm{M}}\right)\mathrm{d}\xi$ in Eq.(1) are generated mainly in the relatively low energy range $v \lesssim v_\mathrm{c}$. This scaling on the **B**-field strength modulation $\propto 1 - \langle B_\mathrm{M}/B\rangle$ is consistent with the fact that reductions of $\left\langle B^{-1}\int x_a^2 L_j^{(5/2)}(x_a^2)P_2(\xi)C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_\mathrm{f}\right)\mathrm{d}^3\mathbf{v}\right\rangle$ caused by $\left\langle \mathbf{b}\cdot\nabla\left(\int_{-1}^1 \xi\overline{f}_\mathrm{f}\mathrm{d}\xi/B\right)\right\rangle$ in Eq.(18) are proportional to the field strength modulation as shown in Fig.2. Although the fast ion source term being $\propto \delta\left(v - v_\mathrm{b}\right)$ in Eq.(56) and in resultant Eq.(61) balances not with the operator $V_\parallel\overline{f}_\mathrm{f}$ but with the collision term $\sum_{b\neq\mathrm{f}} C_\mathrm{fb}\left(f_\mathrm{f}, f_{b\mathrm{M}}\right)$ only, as a result of this balance, poloidal and toroidal variations of the source term make analogous variations of the collision term at the slowing down energy range $v < v_\mathrm{b}$ (mainly $v < v_\mathrm{c}$) that balance with $V_\parallel\overline{f}_\mathrm{f}$. Eq.(61) includes this effect of the source term. When this equation is calculated for the friction collisions between the fast and the thermalized ions $a \neq$ e,f, this source term contribution is only a minor component in the $\mathbf{B}\cdot\nabla\left\{\int \xi G\left(x_a\right)\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\right\}$. For friction collisions between the electrons and the fast ions $a =$ e, the first term in Eq.(61) is very small because of the particle conservation $\int\sum\limits_{b\neq\mathrm{f}} C_{\mathrm{f}b}\left(f_{\mathrm{f}}, f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v} = 0$, and thus the second term as the source term effect is dominant. Next, since the adjoint equation method gives only the surface-averaged quantity $\left\langle\mathbf{b}\cdot\nabla\left\{\int \xi G(x_a)\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\right\}\right\rangle$, the comparison of Eqs.(56) and (57) requires an assumption on the real space structure of $\sum\limits_{b\neq\mathrm{f}} C_{\mathrm{f}b}\left(\overline{f}_{\mathrm{f}}^{(l=0)}, f_{b\mathrm{M}}\right)$. When this collision term is generated at the low energy range $v \lesssim v_{\mathrm{c}}$ as discussed above, because of the "parallel force balance"that was mentioned at the end of Sec.II,

$$\mathbf{B}\cdot\nabla\frac{\int \xi G(x_a)\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}}{B} \simeq \left\langle \mathbf{b}\cdot\nabla\frac{\int \xi G(x_a)\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}}{B}\right\rangle \left\langle\left(\frac{1}{B}-\left\langle\frac{1}{B}\right\rangle\right)^2\right\rangle^{-1}\left(\frac{1}{B}-\left\langle\frac{1}{B}\right\rangle\right),$$

$$\sqrt{\left\langle\left(\mathbf{B}\cdot\nabla\frac{\int \xi G(x_a)\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}}{B}\right)^2\right\rangle} \simeq \left|\left\langle \mathbf{b}\cdot\nabla\frac{\int \xi G(x_a)\overline{f}_{\mathrm{f}}\mathrm{d}^3\mathbf{v}}{B}\right\rangle\right| \left\langle\left(\frac{1}{B}-\left\langle\frac{1}{B}\right\rangle\right)^2\right\rangle^{-1/2} \tag{62}$$

will be appropriate. Then this quantity is compared with the radial gradient effect in Eq.(57). For a rough estimation, the second Legendre order moment $\left\langle\int_{-1}^{1} P_2(\xi)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle$ in $\frac{\partial}{\partial s}\left\langle\int_{-1}^{1}\left(1+\xi^2\right)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle = \frac{4}{3}\frac{\partial}{\partial s}\left\langle\int_{-1}^{1}\left(1+\frac{1}{2}P_2(\xi)\right)\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle$ is neglected. The $\int_0^\infty G(x_a)v^3\mathrm{d}v$ integral of this radial gradient is estimated by

$$\begin{aligned}&\int_0^\infty G(x_a)v^3\frac{\partial\left\langle\int_{-1}^{1}\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle}{\partial r}\mathrm{d}v\\ &\simeq\begin{cases}\dfrac{T_a}{m_a}\dfrac{\partial}{\partial r}\displaystyle\int_0^\infty v\left\langle\int_{-1}^{1}\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle\mathrm{d}v \approx -\dfrac{T_a}{m_a}\dfrac{\langle B\rangle}{\mathrm{d}\psi/\mathrm{d}r}\displaystyle\int_0^\infty v\left\langle\int_{-1}^{1}\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle\mathrm{d}v & \text{for ions } a\neq \text{e,f}\\ \dfrac{2}{3\sqrt{\pi}v_{\mathrm{Te}}}\dfrac{\partial}{\partial r}\displaystyle\int_0^\infty v^4\left\langle\int_{-1}^{1}\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle\mathrm{d}v \approx -\dfrac{2}{3\sqrt{\pi}v_{\mathrm{Te}}}\dfrac{\langle B\rangle}{\mathrm{d}\psi/\mathrm{d}r}\displaystyle\int_0^\infty v^4\left\langle\int_{-1}^{1}\overline{f}_{\mathrm{f}}\mathrm{d}\xi\right\rangle\mathrm{d}v & \text{for electrons } a=\text{e}\end{cases}\end{aligned} \tag{63}$$

with the minor radius r that can be obtained by analytical integral formulas as already mentioned on Eq.(8). The RMS of the (θ,ζ) space oscillation is calculated by

$$\begin{aligned}&\sqrt{\left\langle\left[\frac{B^2}{\langle B^2\rangle}\left\{\sum_{(m,n)\neq(0,0)}\left(B_\zeta^{(\mathrm{Boozer})}m+B_\theta^{(\mathrm{Boozer})}n\right)\varepsilon_{mn}^{(\mathrm{Boozer})}\sin(m\theta_{\mathrm{B}}-n\zeta_{\mathrm{B}})\right\}\right]^2\right\rangle}\\ &\cong\sqrt{\frac{1}{2}\sum_{(m,n)\neq(0,0)}\left\{\left(B_\zeta^{(\mathrm{Boozer})}m+B_\theta^{(\mathrm{Boozer})}n\right)\varepsilon_{mn}^{(\mathrm{Boozer})}\right\}^2}.\end{aligned} \tag{64}$$

By this method, we can confirm that the contribution of the slowing down collision in Eq.(56) is smaller than the radial gradient effect in Eq.(57) by factors of $10^{-5} \sim 10^{-4}$ in conditions assumed in Refs.6 and 9 and this paper. In spite of this large value of the diamagnetic flux divergence effect, it is irrelative to the surface-averaged DKE term balances such as Eqs.(18) and (61) in configurations with the stellarator symmetry $B(-\theta,-\zeta) = B(\theta,\zeta)$ within an accuracy neglecting $\partial \langle n_{\rm f}\mathbf{u}_{\rm f}\cdot\nabla V\rangle/\partial V$ in the particle balance and $\partial\langle\mathbf{Q}_{\rm f}\cdot\nabla V\rangle/\partial V$ in the energy balance for the fast ions themselves. Because of this difference between the two types of the parallel flow generations, the procedure in Sec.II neglects this diamagnetic (perpendicular gradient) effect and includes only Eq.(56) due to the slowing down collision, while we must include $\int_{-1}^{1}\xi C_{a\rm f}\left(f_{a\rm M},\overline{f}_{\rm f1}\right)\mathrm{d}\xi - \left\langle B\int_{-1}^{1}\xi C_{a\rm f}\left(f_{a\rm M},\overline{f}_{\rm f1}\right)\mathrm{d}\xi\right\rangle B/\langle B^2\rangle \propto \widetilde{U}$ determined by Eq.(49) and can neglect Eq.(56) in the DKEs for thermalized particle species $a \neq$ f (the P-S part in Sec.III-B).This estimation method for the $\sum_{b\neq\rm f} C_{{\rm f}b}\left(\overline{f}_{\rm f} - \left\langle\overline{f}_{\rm f}^{(l=0)}\right\rangle, f_{b\rm M}\right)$ may be applicable also for the poloidal/toroidal variation of $\int L_j^{(1/2)}(x_a^2)C_{a\rm f}\left(f_{a\rm M}, f_{\rm f}\right)\mathrm{d}^3\mathbf{v}$ for the thermalized particles that can be included in Eq.(28) together with the Fourier-Laguerre series expressions of $\sum_{b\neq\rm f}\left[C_{ab}\left(h_a^{(l=0)}, f_{b\rm M}\right) + C_{ab}\left(f_{a\rm M}, h_b^{(l=0)}\right)\right]$. However, this variation is smaller than the contribution of $\left\langle\overline{f}_{\rm f}^{(l=0)}\right\rangle$ in Eq.(A6) that should be removed in Eq.(21). Therefore, in Eq.(28) for the purpose of the parallel friction collision, the $\int L_j^{(1/2)}(x_a^2)C_{a\rm f}\left(f_{a\rm M}, f_{\rm f}\right)\mathrm{d}^3\mathbf{v}$ also is neglected together with $\mathbf{B}\cdot\nabla\left(\int_{-1}^{1}\xi\overline{f}_{\rm f}\mathrm{d}\xi/B\right)$ given by Eq.(56).

Next, the perpendicular flow moments $\int \mathbf{v}_\perp v^{k-2}\widetilde{f}_{\rm f}\mathrm{d}^3\mathbf{v}$ with $k = -1, 1, 2, 4, 6$ of the gyrophase-dependent part of the fast ion velocity distribution for the perpendicular friction integrals in the classical diffusions must be considered. For consistency in the construction of the flux-surface coordinates systems that is mentioned in the introduction (i.e., $\nabla\cdot\mathbf{B} = \nabla\cdot\mathbf{J} = \mathbf{B}\cdot\nabla s = \mathbf{J}\cdot\nabla s = 0$ is assumed), we shall derive the perpendicular particle flux $n_{\rm f}\mathbf{u}_{\perp\rm f} \equiv \int \mathbf{v}_\perp \widetilde{f}_{\rm f}\mathrm{d}^3\mathbf{v}$ as a component of the perpendicular current by using a component of $\nabla\cdot(p_{\rm f}\mathbf{I} + \boldsymbol{\pi}_{\rm f})$ that satisfies $\nabla\cdot(n_{\rm f}\mathbf{u}_{\rm f}) = 0$ and $n_{\rm f}\mathbf{u}_{\rm f}\cdot\nabla s = 0$. It also should be noted that the surface-averaged radial transport flux $\langle n_{\rm f}\mathbf{u}_{\rm f}\cdot\nabla s\rangle$ is excluded in calculations of the friction integrals $\int \mathbf{v}_\perp L_j^{(3/2)}(x_a^2)C_{a\rm f}\left(f_{a\rm M}, f_{\rm f}\right)\mathrm{d}^3\mathbf{v}$, $\int \mathbf{v}_\perp L_j^{(3/2)}(x_a^2)C_{a\rm f}\left(f_{a\rm M}, f_{\rm f}\right)\mathrm{d}^3\mathbf{v}$ and the current $\mathbf{J}_\perp$.

Therefore the radial pressure gradient

$$
\begin{aligned}
&(\nabla s)\cdot\nabla\cdot(p_{\rm f}\mathbf{I}+\boldsymbol{\pi}_{\rm f})\\
&\cong|\nabla s|^2\left[\frac{\partial\left\langle p_{\rm f}\right\rangle}{\partial s}-\frac{1}{6}\frac{\partial}{\partial s}\left(\left\langle\frac{p_{\perp\rm f}-p_{\|\rm f}}{B}\right\rangle/\left\langle B^{-1}\right\rangle\right)+\frac{B^2}{2}\frac{\partial}{\partial s}\left(\left\langle\frac{p_{\perp\rm f}-p_{\|\rm f}}{B}\right\rangle\left\langle B^{-1}\right\rangle\right)\right]
\end{aligned}
$$

is used for the perpendicular particle flux. The friction integrals also are obtained by using $\int\mathbf{v}_\perp v^{k-2}\widetilde{f}_{\rm f}{\rm d}^3\mathbf{v}$ driven by $\frac{\partial}{\partial s}\left(\left\langle\int v^kP_2(\xi)f_{\rm f}{\rm d}^3\mathbf{v}/B\right\rangle/\left\langle B^{-1}\right\rangle\right)$ and $\frac{\partial}{\partial s}\left(\left\langle\int v^kP_2(\xi)f_{\rm f}{\rm d}^3\mathbf{v}/B\right\rangle\left\langle B^{-1}\right\rangle\right)$ with $k=-1,1,2,4,6$, which are generalizations of the method for the current. In contrast to the parallel flow moments $\int\mathbf{v}_\| v^{k-2}\overline{f}_{\rm f1}{\rm d}^3\mathbf{v}$, here we should include the field curvature effect $\mathbf{b}\cdot\nabla\mathbf{b}\cong\nabla_\perp\ln B$ in the CGL tensor formula that we did not handle explicitly for $\nabla_\perp\overline{f}_{\rm f}$ in the fast ion DKE. The perpendicular component of the $\int\mathbf{v}v^{k-2}{\rm d}^3\mathbf{v}$ integral of the full Vlasov operator for the fast ions gives

$$
\begin{aligned}
&\int\mathbf{v}_\perp v^{k-2}\widetilde{f}_{\rm f}{\rm d}^3\mathbf{v}=-\frac{m_{\rm f}c}{e_{\rm f}}\frac{1}{B^2}\left(\nabla\cdot\int v^{k-2}\mathbf{v}\mathbf{v}f_{\rm f}{\rm d}^3\mathbf{v}\right)\times\mathbf{B}\\
&=-\frac{1}{3}\frac{m_{\rm f}c}{e_{\rm f}}\frac{1}{B^2}\nabla\left(\int v^kf_{\rm f}{\rm d}^3\mathbf{v}\right)\times\mathbf{B}-\frac{m_{\rm f}c}{e_{\rm f}}\frac{1}{B^2}\nabla\cdot\left\{(\mathbf{bb}-\mathbf{I}/3)\int v^kP_2(\xi)f_{\rm f}{\rm d}^3\mathbf{v}\right\}\times\mathbf{B}\\
&=-\frac{1}{3}\frac{m_{\rm f}c}{e_{\rm f}}\frac{1}{B^2}\nabla\left(\int v^kf_{\rm f}{\rm d}^3\mathbf{v}\right)\times\mathbf{B}-\frac{1}{6}\frac{m_{\rm f}c}{e_{\rm f}}\frac{1}{B^2}\nabla\left(\int v^kP_2(\xi)f_{\rm f}{\rm d}^3\mathbf{v}\right)\times\mathbf{B}\\
&\qquad+\frac{1}{2}\frac{m_{\rm f}c}{e_{\rm f}}\nabla\frac{\int v^kP_2(\xi)f_{\rm f}{\rm d}^3\mathbf{v}}{B^2}\times\mathbf{B}.
\end{aligned}
\tag{65}
$$

In this determination of $\int\mathbf{v}_\perp v^{k-2}\widetilde{f}_{\rm f}{\rm d}^3\mathbf{v}$ vectors, the perpendicular friction integrals $\int\mathbf{v}_\perp v^{k-2}\sum_b C_{\rm f b}\left(f_{\rm f},f_b\right){\rm d}^3\mathbf{v}$ are neglected since $e_{\rm f}c^{-1}B/m_{\rm f}\gg 1/\tau_{\rm S}$, and the perpendicular electric field term $(\nabla_\perp\Phi)\cdot\partial f_{\rm f}/\partial\mathbf{v}$ also is neglected by a reason noted on Eq.(1). These perpendicular gradient forces are given by a generalization of usual formulas for the CGL tensors $\boldsymbol{\pi}_a=\left(p_{\|a}-p_{\perp a}\right)(\mathbf{bb}-\mathbf{I}/3)$ and $\mathbf{r}_a-r_a\mathbf{I}=\left(r_{\|a}-r_{\perp a}\right)(\mathbf{bb}-\mathbf{I}/3)$ to $k=-1,1,6$. This CGL method corresponding to a replacement $\mathbf{vv}\cong v^2\mathbf{I}/3+\left(v_\|^2-v_\perp^2/2\right)(\mathbf{bb}-\mathbf{I}/3)=v^2\mathbf{I}/3+v^2P_2(\xi)(\mathbf{bb}-\mathbf{I}/3)$ neglects a possibility of non-diagonal elements in the tensors $\int v^{k-2}\mathbf{vv}f_{\rm f}{\rm d}^3\mathbf{v}$. Although the co-existence of the large first Legendre order in the gyro-phase-averaged part $\int_{-1}^1\xi\overline{f}_{\rm f}{\rm d}\xi$ and the gyro-phase-dependent part $\widetilde{f}_{\rm f}\cong\frac{m_{\rm f}c}{e_{\rm f}B}\mathbf{v}\cdot\left(\mathbf{b}\times\nabla\overline{f}_{\rm f}\right)$ may generate these non-diagonal elements (since $\xi P_1^1(\xi)\cos\phi=\frac{1}{3}P_2^1(\xi)\cos\phi$), this method for $\nabla\cdot\int v^{k-2}\mathbf{vv}f_{\rm f}{\rm d}^3\mathbf{v}$ with $k\geq 1$, where the full energy range $0\leq v\leq v_{\rm b}$ contribute to this integral, is justified by the fact that the inertia force $\propto\nabla\cdot\left\{n_{\rm f}\left(\mathbf{u}_{\rm f}\mathbf{u}_{\rm f}-\mathbf{u}_{\|\rm f}\mathbf{u}_{\|\rm f}\right)\right\}=n_{\rm f}\left(\mathbf{u}_{\rm f}\cdot\nabla\mathbf{u}_{\rm f}-\mathbf{u}_{\|\rm f}\cdot\nabla\mathbf{u}_{\|\rm f}\right)+\mathbf{u}_{\|\rm f}\nabla\cdot\left(n_{\rm f}\mathbf{u}_{\perp\rm f}\right)$ is negligible in comparison with the pressure gradient $\nabla p_{\rm f}$ even in the unbalanced tangential NBI operations giving $m_{\rm f}n_{\rm f}u_{\|\rm f}^2\sim p_{\rm f}$. It also should be noted that the "parallel pressure"in this tensor calculation is defined by $p_{\|\mathrm{f}} \equiv m_{\mathrm{f}} \int v_{\|}^{2} f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}$ as in Sec.II and Appendix A (not $m_{\mathrm{f}} \int \left|\mathbf{v}_{\|} - \mathbf{u}_{\|\mathrm{f}}\right|^{2} f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}$) for including the parallel flow curvature effect $n_{\mathrm{f}} \mathbf{u}_{\|\mathrm{f}} \cdot \nabla \mathbf{u}_{\|\mathrm{f}}$ in the viscosity tensor term $\nabla \cdot \boldsymbol{\pi}_{\mathrm{f}}$ instead of the explicit calculation of the flow vector field $n_{\mathrm{f}} \mathbf{u}_{\|\mathrm{f}}$ in this force term $\nabla \cdot \int \mathbf{v}\mathbf{v} f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}$. In the extension of this method to $k = -1$ where $\nabla \cdot \int v^{-3} \mathbf{v}\mathbf{v} f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}$ must be handled, we should recall that the high-energy region $v > v_{\mathrm{c}}$ of both the gyro-phase-averaged distribution $\overline{f}_{\mathrm{f}}$ and the gyro-phase-dependent component $\widetilde{f}_{\mathrm{f}}$ cannot effectively contribute to the integral $\int v^{-3} \mathbf{v}\mathbf{v} f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}$. The neglect of the possibility of the non-diagonal elements in it is justified for situations of $\left|\int \mathbf{v}_{\perp} v^{-3} \widetilde{f}_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}\right| \left|\int \mathbf{v}_{\|} v^{-3} \overline{f}_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}\right| \ll \left(\int v^{-2} f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}\right)^{2}$, and this condition is satisfied when the radial gradient scale length is $\frac{m_{\mathrm{f}} c}{e_{\mathrm{f}} B} v_{\mathrm{c}} \ll \left|\nabla \ln \int v^{-2} f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}\right|^{-1}$. As a result of the field curvature $\mathbf{b} \cdot \nabla \mathbf{b} = \nabla_{\perp} \ln B$, the perpendicular flow integrals $\int \mathbf{v}_{\perp} v^{k-2} \widetilde{f}_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}$ consist of two types of contributions of the anisotropy. One is $\nabla \left(\left\langle \int v^{k} P_{2}(\xi) f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}/B \right\rangle / \left\langle B^{-1} \right\rangle\right) \times \mathbf{B}/B^{2}$ that satisfies $\nabla \cdot \int \mathbf{v} v^{k-2} f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v} = 0$ as a sum of this perpendicular component and the parallel component in Eq.(50), and the other is $\nabla \left(\left\langle \int v^{k} P_{2}(\xi) f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}/B \right\rangle \left\langle B^{-1} \right\rangle\right) \times \mathbf{B}$ that does not cause the divergence.

Therefore the inclusion of the fast ions' anisotropy in the P-S and the classical diffusions and the parallel and the perpendicular currents requires only $\left\langle \int v^{k} P_{2}(\xi) f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}/B \right\rangle$ with $k = -1, 1, 2, 4, 6$. Here we show some numerical examples on the configuration dependences of these quantities in Fig.4. The assumed magnetic field and the plasma parameters are those in Fig.2 in Sec.III-C. Also in this figure, the ratios $\left\langle \int v^{k} P_{2}(\xi) \overline{f}_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}/B \right\rangle / \left\langle \int v^{k} P_{2}(\xi) \overline{f}_{\mathrm{f}}^{(\mathbf{b}\cdot\nabla B=0)} \mathrm{d}^{3}\mathbf{v}/B \right\rangle$ are shown. The collision integral $\left\langle \int x_{a}^{2} P_{2}(\xi) C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right) \mathrm{d}^{3}\mathbf{v}/B \right\rangle$ in Fig.2 and the velocity distribution integral $\left\langle \int v^{-1} P_{2}(\xi) f_{\mathrm{f}} \mathrm{d}^{3}\mathbf{v}/B \right\rangle$ in Fig.4 have analogous configuration dependence since they are obtained by an analogous energy space weighting. The deviation of the ratios from the unity is proportional to $1 - \langle B \rangle / B_{\mathrm{M}}$. Although this scaling on the $\mathbf{B}$-field strength is different from that of the previously investigated parallel momentum input, the dependence on the energy space weighting v^{k} indicates an analogous physical process. The reason for this scaling on the field strength modulation has already been stated in previous sections. In the tangential NBIs, this reduction of $\int_{-1}^{1} \xi \overline{f}_{\mathrm{f}} \mathrm{d}\xi$ and/or $\int_{-1}^{1} P_{2}(\xi) \overline{f}_{\mathrm{f}} \mathrm{d}\xi$ caused by the parallel guding center motion conserving the magnetic moment is important for a slow velocity range $v \lesssim v_{\mathrm{c}}$ where the velocity distribution is broadened in the pitch-angle space, and is unimportant for the high energy range where the velocity distribution is still localized

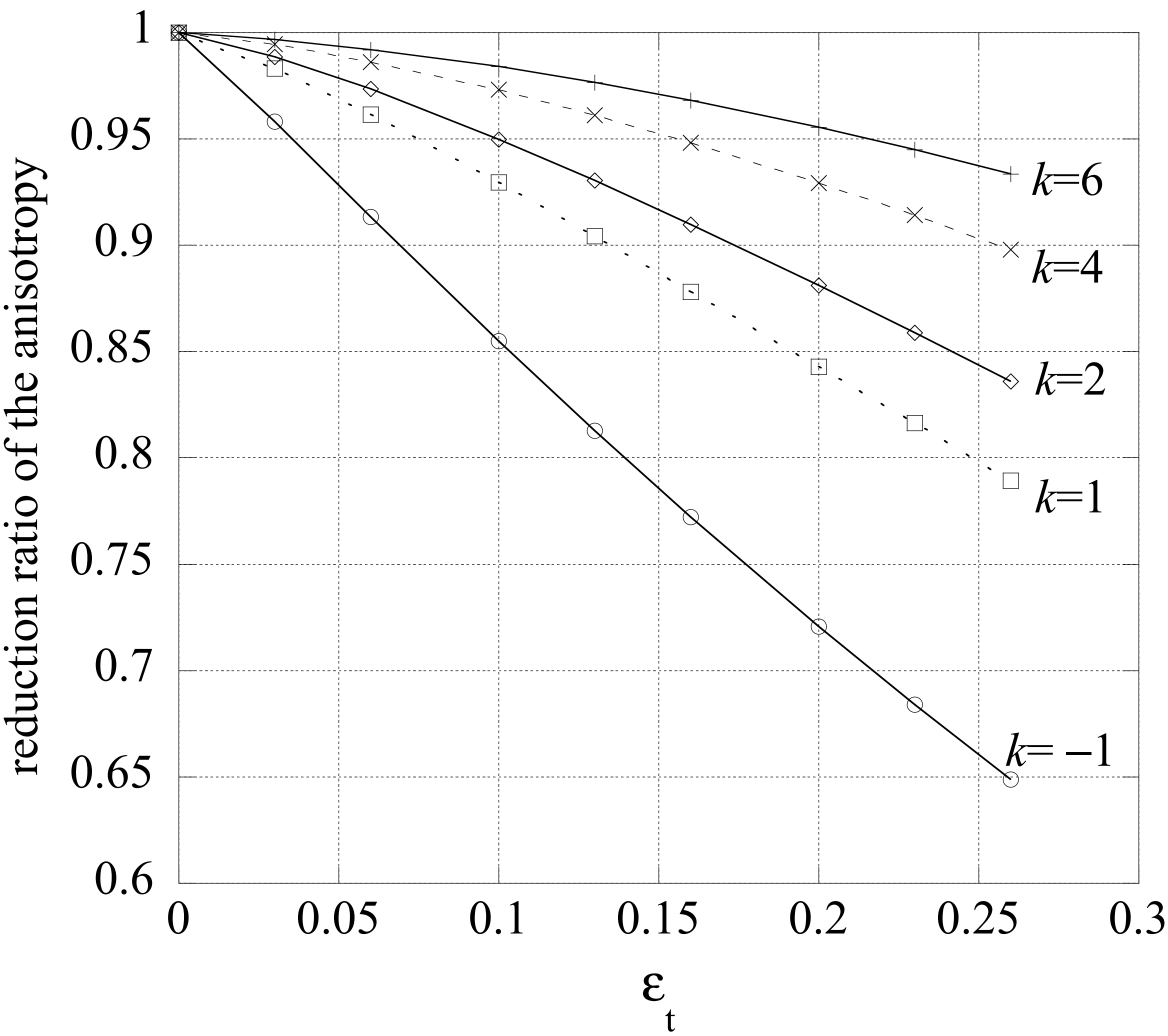

FIG. 4. The configuration dependence of the $\left\langle \int v^k P_2(\xi) f_{\rm f} {\rm d}^3\mathbf{v}/B \right\rangle$ integrals for the model field Eq.(47). The result is normalized by the calculation using Eq.(20) for $\mathbf{b}\cdot\nabla B = 0$.

around the injection pitch-angle. This characteristic of $\overline{f}_{\rm f}(\mathbf{x}, v, \sigma, \lambda)$ gives the dependence on v^k in Fig.4.

Finally, for the radial gradient of the lowest Legendre order $\partial \left\langle \overline{f}_{\rm f}^{(l=0)} \right\rangle / \partial s$ in both Eq.(50) and the gyro-phase-dependent component of the distribution $\frac{m_{\rm f} c}{e_{\rm f} B} \mathbf{v}\cdot(\mathbf{b}\times\nabla s)\, \partial \left\langle \overline{f}_{\rm f}^{(l=0)} \right\rangle / \partial s$ that should be directly substituted into the friction integral formulas $\int \mathbf{v} L_j^{(3/2)}(x_a^2) C_{a{\rm f}}\left(f_{a{\rm M}}, f_{\rm f}\right) {\rm d}^3\mathbf{v}$ and $\int \mathbf{v} v^{2j} \sum_{b\neq{\rm f}} C_{{\rm f}b}\left(f_{\rm f}, f_{b{\rm M}}\right) {\rm d}^3\mathbf{v}$, the aforementioned analytical solution is used. The radial gradient is given by

$$\frac{1}{2}\frac{\partial}{\partial s}\frac{S_0\tau_{\mathrm{S}}}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3}H\left(v_{\mathrm{b}}-v\right)$$
$$=\frac{1}{2}\frac{S_0\tau_{\mathrm{S}}}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3}H\left(v_{\mathrm{b}}-v\right)\times$$
$$\left[\frac{\partial\ln\left(S_0\tau_{\mathrm{S}}\right)}{\partial s}-\frac{\partial\ln v_{\mathrm{c}}}{\partial s}\frac{3v_{\mathrm{c}}^3}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3}-\frac{\partial\ln v_{\mathrm{Te}}}{\partial s}\frac{3}{2}v_{\mathrm{Te}}v^2\frac{3\sqrt{\pi}G\left(x_{\mathrm{e}}\right)-2x_{\mathrm{e}}\exp\left(-x_{\mathrm{e}}^2\right)}{v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G\left(x_{\mathrm{e}}\right)+v_{\mathrm{c}}^3}\right].$$

Within the aforementioned accuracy neglecting $\partial\left\langle n_{\mathrm{f}}\mathbf{u}_{\mathrm{f}}\cdot\nabla V\right\rangle/\partial V$ and $\partial\left\langle \mathbf{Q}_{\mathrm{f}}\cdot\nabla V\right\rangle/\partial V+e_{\mathrm{f}}\left\langle n_{\mathrm{f}}\mathbf{u}_{\mathrm{f}}\cdot\nabla V\right\rangle\partial\Phi/\partial V$, we do not need to consider the dependence of this $\left\langle\overline{f}_{\mathrm{f}}^{(l=0)}\right\rangle$ on the field strength modulation $B\left(\theta,\zeta\right)$ on the flux-surfaces.

V. DISCUSSION

Since this work was conducted assuming the tangential NBI operations, analogous to Refs.9, 20, and 21, the 0th order of $\rho_{\mathrm{f}}/L_r \propto \langle B\rangle^{-1}$ in the fast ion velocity distribution was investigated based on Eq.(1) in which the perpendicular guiding center velocity $\mathbf{v}_{\mathrm{df}}=e_{\mathrm{f}}^{-1}c\left(m_{\mathrm{f}}v_{\|}^2/B+\mu\right)\mathbf{b}\times\nabla\mathrm{ln}B$ is excluded. In other different injection conditions causing the beam ionization at the deeply trapped pitch-angle range,[34] together with the radial drift velocity $\mathbf{v}_{\mathrm{df}}\cdot\nabla s$ that is investigated in Sec.IV, the poloidal precession of the deeply trapped particles in $\kappa^2\leq 1$ due to $\partial\varepsilon_{\mathrm{H}}/\partial s$ and $\partial\varepsilon_{\mathrm{T}}/\partial s$ also will be required.[25] In this injection method, however, the injection energy will be chosen to be low. As a result of the strong pitch-angle-scattering collision in the slow velocity range $v<v_{\mathrm{c}}$, the anisotropy cannot become so large. Rather than this situation, the present study is focused on tangential NBI operations with the injection energies of $v_{\mathrm{c}}^2<v_{\mathrm{b}}^2\lesssim 2T_{\mathrm{e}}/m_{\mathrm{e}}$ where the contribution of the high-energy range $v>v_{\mathrm{c}}$ of the fast ions' velocity distribution, for which the slowing down is the dominant collision process, to the pressure anisotropy $p_{\|\mathrm{f}}>p_{\perp\mathrm{f}}$ is large. In the low energy range, this velocity distribution is broadened to the full pitch-angle range because of the PAS collision, and the calculation of the anisotropy (the second Legendre order) requires this full range. For the purpose of the surface-averaged velocity space integrals such as integrals with a form of $\left\langle\int H_2(v)P_2(\xi)f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}/B\right\rangle$ in the tangential NBIs, however, only the solution of the adjoint equation at the circulating pitch-angle range $0\leq\lambda\leq 1$ is required. The $-c\nabla\Phi\times\mathbf{B}/B^2$ precession and the collisionless detrapping/retrapping of the low-energy trapped particles in $\kappa^2 \leq 1$ are implicitly allowed in this adjoint equation method. In actual experimental conditions, the velocity space loss region[25] is often eliminated by this mechanism. This is another reason for which the direct solving for the DKE for $\overline{f}_{\rm f}^{(\rm even)}$ itself is difficult. The result of the adjoint equation method for the $\left\langle \int H_2(v)P_2(\xi)f_{\rm f}{\rm d}^3\mathbf{v}/B\right\rangle$ integrals includes the previously known analytical solution in Ref.28 as a limit of $B = B_{\rm M}$. The deviation of the actual integrals from this limit was proportional to $1 - \langle B\rangle / B_{\rm M}$. This is a contrasting appearance of the parallel guding center motion effect compared with that in the previously investigated parallel momentum input (especially that for thermalized ions)[9] where the deviation is almost determined by $\int_0^1 \lambda \left\langle (1 - \lambda B/B_{\rm M})^{1/2}\right\rangle^{-1} {\rm d}\lambda$. (Note also that the comparisons shown in this paper are those of the results of the adjoint equation method for including $\left\langle B^{-1}\left(\int_{-1}^{1}\xi\overline{f}_{\rm f}{\rm d}\xi\right)\mathbf{b}\cdot\nabla{\rm ln}B\right\rangle$ in Eq.(18) and the results of Eq.(20) where this term is neglected.) This difference is caused by the fact that the full pitch-angle range $0 \leq \lambda \leq B_{\rm M}/B$ determines the anisotropy, while the momentum input is determined only by the circulating pitch-angle range $0 \leq \lambda \leq 1$.

After this derivation of formulas for the $\left\langle \int H_2(v)P_2(\xi)f_{\rm f}{\rm d}^3\mathbf{v}/B\right\rangle$ integrals in Sec.II, we applied this method for the anisotropic heating analysis of the thermalized target plasma species in Sec.III. The handling of the newly added DKE term $C_{a{\rm f}}\left(f_{a{\rm M}}, f_{\rm f}^{(l=2)}\right)$ is independent of the previously investigated parts for handling the radial gradient forces $\frac{\partial}{\partial s}\langle p_a\rangle$, $\frac{\partial}{\partial s}\langle r_a\rangle$, $\frac{\partial}{\partial s}\Phi$ and the parallel force terms $E_{\|}^{(\mathbf{A})}$, $C_{a{\rm f}}\left(f_{a{\rm M}}, f_{\rm f}^{(l=1)}\right)$, and thus a precedent calculation is possible. In contrast to the parallel momentum input $\left\langle B\int_{-1}^{1}\xi C_{a{\rm f}}\left(f_{a{\rm M}}, \overline{f}_{\rm f}\right){\rm d}\xi\right\rangle$ causing $\left\langle \mathbf{\Gamma}_a^{\rm bn}\cdot\nabla s\right\rangle$, $\left\langle \mathbf{Q}_a^{\rm bn}\cdot\nabla s\right\rangle$, and $\langle \mathbf{J}\cdot\mathbf{B}\rangle$, and to the poloidal and the toroidal variations of the friction $\int_{-1}^{1}\xi C_{a{\rm f}}\left(f_{a{\rm M}}, \overline{f}_{\rm f}\right){\rm d}\xi - \left\langle B\int_{-1}^{1}\xi C_{a{\rm f}}\left(f_{a{\rm M}}, \overline{f}_{\rm f}\right){\rm d}\xi\right\rangle B/\langle B^2\rangle \propto \widetilde{U}$ being a direct contribution to $\left\langle \mathbf{\Gamma}_a^{\rm PS}\cdot\nabla s\right\rangle$ and $\left\langle \mathbf{Q}_a^{\rm PS}\cdot\nabla s\right\rangle$, this newly added source term does not generate velocity distribution components nor collision terms concerning these transport fluxes. Another contrasting situation in this anisotropic heating analysis is that the algebraic conversion of the DKE as an integro-differential equation including the full linearized collision operator for the second Legendre order $l = 2$ does not require any contribution of the operator $V_{\|}$. When solving the parallel force balance equation in the previously investigated part, the parallel viscosity matrix $M_{j+1,k+1}^a$ for the parallel flow moments was essentially required together with the friction matrix $l_{j+1,k+1}^{ab}$ since the latter matrix does not have its inverse matrix because of the momentum conserving and Galilean invariant property of the Coulomb collision. This viscosity matrix corresponded to a part of the $V_{\|}$ operator. In the anisotropic heating analysis, this kind of role of the $V_{\|}$ is not essential since the collisional anisotropy relaxation matrix (Appendix B) has its inverse matrix. As a result of this inverse matrix and the $\int x_a^2 P_2(\xi) L_j^{(5/2)}(x_a^2) C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}})\, \mathrm{d}^3\mathbf{v}$ integrals given by the adjoint equation method in Sec.II, it is found that the fast-ion-driven anisotropies are $\left\langle (p_{\|\mathrm{e}} - p_{\perp\mathrm{e}})/B \right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle \approx 3 \times 10^{-5}$ for electrons ($Z_{\mathrm{eff}} = 1.9$) and $(\langle p_{\mathrm{e}} \rangle / \langle p_a \rangle) \left| \left\langle (p_{\|a} - p_{\perp a})/B \right\rangle / \left\langle (p_{\|\mathrm{f}} - p_{\perp\mathrm{f}})/B \right\rangle \right| < 10^{-2}$ for thermalized ions $a \neq$ e,f (even when $v_{\mathrm{b}} \sim v_{\mathrm{c}}$) in typical operational conditions (assumed in Refs.6 and 9). Although these ratios must be confirmed for individual experimental conditions, if these small values are obtained, these thermalized species can be regarded as the "isotropic-pressure"species in all of the parallel and perpendicular currents in the MHD equilibrium, the radial gradient term $(\mathbf{v}_{\mathrm{d}a} \cdot \nabla s)\, \partial f_{a\mathrm{M}}/\partial s$ in the DKEs, and the classical diffusions. Even though a relation $C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}}) \cong C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l\leq 1)}\right)$ is not generally guaranteed for the NB-produced fast ions, the response to $C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}^{(l\geq 2)}\right)$ can be neglected in these radial gradient calculations in many practical cases.

In Sec.IV, the adjoint equation method was applied for the perpendicular friction and the poloidal/toroidal variations of the parallel friction causing the classical and the P-S radial diffusions of both the thermalized target plasma species and the fast ions themselves. Basically, the friction integrals $\int \mathbf{v} v^{2j} \sum_{a\neq \mathrm{f}} C_{\mathrm{f}a}(f_{\mathrm{f}}, f_{a\mathrm{M}}) \mathrm{d}^3\mathbf{v}$ $(j = 0, 1)$ and $\int \mathbf{v} L_j^{(3/2)}(x_a^2) C_{a\mathrm{f}}(f_{a\mathrm{M}}, f_{\mathrm{f}})\, \mathrm{d}^3\mathbf{v}$ $(j \leq 2)$ including the fast ion velocity distribution $f_{\mathrm{f}}(\mathbf{x}, \mathbf{v})$ must be obtained by substituting the distribution function directly into these integral formulas explained in Refs.9 and 19 since the conventional methods for thermal-thermal collisions such as the Braginskii's matrix expression cannot be applicable for this function. The radial gradient of the lowest Legendre order $\partial \left\langle f_{\mathrm{f}}^{(l=0)} \right\rangle / \partial s$ can be handled by this direct substituting, since an analytical expression of it that is applicable for general toroidal configurations is already known. However, the adjoint equation method adopted for the anisotropy is a method to obtain appropriate $\int_0^\infty \mathrm{d}v$ integrals instead of the energy space structure of the radial gradient $\frac{\partial}{\partial s} \left\langle \int_{-1}^{1} P_2(\xi) \overline{f}_{\mathrm{f}} \mathrm{d}\xi / B \right\rangle$. Therefore, we shall apply this method for obtaining numerical radial differentials $\frac{\partial}{\partial s} \left\langle \int v^k P_2(\xi) f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}/B \right\rangle$ with $k = -1, 1, 2, 4, 6$. The P-S and the classical diffusions are obtained by summing these differentials, and the fast ions' particle flux $n_{\mathrm{f}}\mathbf{u}_{\mathrm{f}} \equiv \int \mathbf{v} f_{\mathrm{f}} \mathrm{d}^3\mathbf{v}$ in the perpendicular and the P-S parallel currents also is obtained as $k = 2$ in this series. Numerical examples for this $\left\langle \int v^k P_2(\xi) f_{\rm f} {\rm d}^3\mathbf{v}/B \right\rangle$ clarified that the absolute value of the aforementioned deviation $\propto 1 - \langle B \rangle / B_{\rm M}$ depends on the energy space weighting v^k, and this dependence indicates that the parallel guididng center motion effect is important in lower energy ranges where the $f_{\rm f}(\mathbf{x}, \mathbf{v})$ is broadened in the pitch-angle space. This dependence on the energy space weighting is analogous to that in the previous momentum input calculation.

In addition to these surface-averaged effects of the anisotropy, the parallel viscosity forces $\langle \mathbf{B} \cdot \nabla \cdot \boldsymbol{\pi}_{\rm f} \rangle = -\left\langle (p_{\|{\rm f}} - p_{\perp{\rm f}}) \mathbf{B} \cdot \nabla \ln B \right\rangle$, $\langle \mathbf{B} \cdot \nabla \cdot \mathbf{r}_{\rm f} \rangle = -\left\langle (r_{\|{\rm f}} - r_{\perp{\rm f}}) \mathbf{B} \cdot \nabla \ln B \right\rangle$ of the fast ions themselves caused by the poloidal and toroidal variations of the anisotropy are the other effect of the anisotropy. The previous investigation on the momentum input by the unbalanced tangential NBIs clarified also the existence of these parallel forces.[9] The poloidally and toroidally varying anisotropy included there will simultaneously generate also the viscosity-driven radial particle/energy transport fluxes $\left\langle \boldsymbol{\Gamma}_{\rm f}^{\rm bn} \cdot \nabla s \right\rangle =$
$-e_{\rm f}^{-1} c \left\langle (p_{\|{\rm f}} - p_{\perp{\rm f}}) \left(\nabla s \times \mathbf{B}/B^2 + \widetilde{U}\mathbf{b} \right) \cdot \nabla \ln B \right\rangle$ and $\left\langle \mathbf{Q}_{\rm f}^{\rm bn} \cdot \nabla s \right\rangle =$
$-e_{\rm f}^{-1} c m_{\rm f} \left\langle (r_{\|{\rm f}} - r_{\perp{\rm f}}) \left(\nabla s \times \mathbf{B}/B^2 + \widetilde{U}\mathbf{b} \right) \cdot \nabla \ln B \right\rangle$ that are defined there. Since these anisotropies $p_{\|{\rm f}} - p_{\perp{\rm f}}$, $r_{\|{\rm f}} - r_{\perp{\rm f}}$ themselves were not determined directly in this previous investigation, an appropriate calculation method for these radial transport fluxes is a future theme. Since these quantities also are definite integrals of the fast ions' velocity distribution in the 4D space (θ, ζ, v, ξ), the adjoint equation method will be a convenient and powerful method also for this purpose. In contrast to the P-S and the classical diffusion being the intrinsically ambipolar transport process, this viscosity-driven transport is non-ambipolar and must be included in the determination of the ambipolar potential by the ambipolar condition $\langle \mathbf{J} \cdot \nabla s \rangle = 0$. Another difference between $\left\langle \left(\boldsymbol{\Gamma}_{\rm f}^{\rm PS} + \boldsymbol{\Gamma}_{\rm f}^{\rm cl} \right) \cdot \nabla s \right\rangle$, $\left\langle \left(\mathbf{Q}_{\rm f}^{\rm PS} + \mathbf{Q}_{\rm f}^{\rm cl} \right) \cdot \nabla s \right\rangle$ and $\left\langle \boldsymbol{\Gamma}_{\rm f}^{\rm bn} \cdot \nabla s \right\rangle$, $\left\langle \mathbf{Q}_{\rm f}^{\rm bn} \cdot \nabla s \right\rangle$ in the tangential NBIs is that the former is the dominant loss at the high-energy range while the latter is substantially generated in a relatively low-energy range $v \lesssim v_{\rm c}$ of the $f_{\rm f}(\mathbf{x}, \mathbf{v})$ since the finite $\mathbf{B}$-field modulation effect causing $\langle \mathbf{B} \cdot \nabla \cdot \boldsymbol{\pi}_{\rm f} \rangle$, $\langle \mathbf{B} \cdot \nabla \cdot \mathbf{r}_{\rm f} \rangle$ is important in that energy range where the $f_{\rm f}(\mathbf{x}, \mathbf{v})$ is broadened to a wide range of the pitch-angle space. The determination of the 0th order of $\rho_{\rm f}/L_r$ based on Eq.(1) is justified when the total particle/energy losses $\partial \langle \boldsymbol{\Gamma}_{\rm f} \cdot \nabla V \rangle / \partial V$, $\partial \langle \mathbf{Q}_{\rm f} \cdot \nabla s \rangle / \partial V$, which are sums of these transport processes with different roles, are small compared with the fast ion source term. These issues will be studied and reported in a separated article.

ACKNOWLEDGMENTS

The author thanks K.Y. Watanabe, Y. Asahi, S. Satake, and B. Huang for discussions suggesting this theme. This work was supported by the LHD project budgets ULPP035, UMPG701, and the NIFS collaboration research program NIFS17KNXN351.

Appendix A: The perpendicular and the parallel particle/energy fluxes of species with non-negligible anisotropies

In this Appendix, we shall summarize the determination of the perpendicular particle/energy fluxes and the resultant parallel flow divergences based on the $\int \mathrm{d}^3\mathbf{v}$ (particle balance), $\int \mathbf{v}\mathrm{d}^3\mathbf{v}$ (force balance), $\int v^2\mathrm{d}^3\mathbf{v}$ (energy balance), and $\int \mathbf{v}v^2\mathrm{d}^3\mathbf{v}$ (energy weighted force balance) integrals of the Vlasov-Fokker-Planck equation analogous to Ref.9. The steady-state ($\partial\left(n_a\mathbf{u}_a\right)/\partial t=0$, $\partial\mathbf{Q}_a/\partial t=0$) force balance equations are given by

$$\nabla\cdot\left(p_a\mathbf{I}+\boldsymbol{\pi}_a\right)-e_an_a\left(\mathbf{E}+\frac{\mathbf{u}_a\times\mathbf{B}}{c}\right)=\mathbf{F}_{a1},\tag{A1}$$

$$\nabla\cdot\mathbf{r}_a-\frac{e_a}{m_a}\left[\mathbf{E}\cdot\left(\frac{5}{2}p_a\mathbf{I}+\boldsymbol{\pi}_a\right)+\frac{\mathbf{Q}_a\times\mathbf{B}}{c}\right]=\mathbf{G}_a\tag{A2}$$

with $n_a\equiv\int f_a\mathrm{d}^3\mathbf{v}$, $\mathbf{Q}_a\equiv(m_a/2)\int\mathbf{v}v^2f_a\mathrm{d}^3\mathbf{v}$, and the CGL tensors $\boldsymbol{\pi}_a=(p_{\|a}-p_{\perp a})\left(\mathbf{bb}-\mathbf{I}/3\right)$ and $\mathbf{r}_a-r_a\mathbf{I}=(r_{\|a}-r_{\perp a})\left(\mathbf{bb}-\mathbf{I}/3\right)$ including $p_{\perp a}\equiv m_a\int\left|\mathbf{v}_\perp-\mathbf{u}_{\perp a}\right|^2f_a\mathrm{d}^3\mathbf{v}/2$, $p_{\|a}\equiv m_a\int v_\|^2f_a\mathrm{d}^3\mathbf{v}$, $p_a\equiv(2p_{\perp a}+p_{\|a})/3$, $r_{\perp a}\equiv(m_a/2)\int v^2v_\perp^2f_a\mathrm{d}^3\mathbf{v}/2$, $r_{\|a}\equiv(m_a/2)\int v^2v_\|^2f_a\mathrm{d}^3\mathbf{v}$, and $r_a\equiv(2r_{\perp a}+r_{\|a})/3$. The perpendicular forces caused by these tensors are

$$\left\{\nabla\cdot\left(p_a\mathbf{I}+\boldsymbol{\pi}_a\right)\right\}_\perp=\frac{1}{2}\left(\nabla_\perp(p_{\perp a}+p_{\|a})+B^2\nabla_\perp\frac{p_{\perp a}-p_{\|a}}{B^2}\right)\tag{A3}$$

and

$$\left(\nabla\cdot\mathbf{r}_a\right)_\perp=\frac{1}{2}\left(\nabla_\perp(r_{\perp a}+r_{\|a})+B^2\nabla_\perp\frac{r_{\perp a}-r_{\|a}}{B^2}\right).\tag{A4}$$

Next, the steady-state ($\partial p_a/\partial t=0$) energy balance equation is

$$\nabla\cdot\mathbf{Q}_a-e_an_a\mathbf{u}_a\cdot\mathbf{E}=\frac{m_a}{2}\int v^2C_a(f_a)\mathrm{d}^3\mathbf{v}+\text{other source/loss terms}\tag{A5}$$

with $C_a(f_a)\equiv\sum_b C_{ab}\left(f_a,f_b\right)$. When using this equation for the purpose of the parallel energy flux divergence $\nabla\cdot\mathbf{Q}_{\|a}$, the surface-averaged component

$$\begin{aligned}&\frac{\partial}{\partial V}\left\langle\mathbf{Q}_a\cdot\nabla V\right\rangle+e_a\left\langle n_a\mathbf{u}_a\cdot\nabla V\right\rangle\frac{\partial\Phi}{\partial V}-e_a\left\langle n_a\mathbf{u}_a\cdot\mathbf{E}^{(\mathbf{A})}\right\rangle\\&\quad=\frac{m_a}{2}\left\langle\int v^2C_a(f_a)\mathrm{d}^3\mathbf{v}\right\rangle+\text{other source/loss terms,}\end{aligned}\tag{A6}$$

which is given by using the Gauss' theorem for the volume V enclosed by the flux-surface $s=\text{const}$, must be separated for the solubility condition $\left\langle \nabla\cdot\mathbf{Q}_{\|a}\right\rangle = \left\langle \mathbf{B}\cdot\nabla\left(Q_{\|a}/B\right)\right\rangle = 0$. Although the source/loss terms and the collision term in this RHS also must be separated from the DKE for thermalized particles (Eq.(21)), this RHS is included in the DKE for the fast ions (Eq.(1)) since the main purpose of this equation is in the balance of the fast ion source term and the slowing down collision term. Nevertheless, the separation of $\partial\left\langle \mathbf{Q}_a\cdot\nabla V\right\rangle/\partial V$ is commonly required for these general DKEs. Analogously, $\left\langle \nabla\cdot(n_a\mathbf{u}_a)\right\rangle = \partial\left\langle n_a\mathbf{u}_a\cdot\nabla V\right\rangle/\partial V$ must be removed in the particle balance for the purpose of the determination of the steady-state gyro-phase-averaged distribution. For the thermalized particles handled by Eq.(21), the heating energy input by the fast ions $\int v^2 C_{a\mathrm{f}}\left(f_{a\mathrm{M}}, f_{\mathrm{f}}\right)\mathrm{d}^3\mathbf{v}$ and/or the electron-ion temperature relaxation $m_{\mathrm{e}}\int v^2 C_{\mathrm{e}a}\left(f_{\mathrm{eM}}, f_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v} = -m_a\int v^2 C_{a\mathrm{e}}\left(f_{a\mathrm{M}}, f_{\mathrm{eM}}\right)\mathrm{d}^3\mathbf{v} = -6n_{\mathrm{e}}\tau_{\mathrm{e}a}^{-1}\left(m_{\mathrm{e}}/m_a\right)\left(T_{\mathrm{e}}-T_a\right)$ for $a\neq$ e,f is included in the RHS of Eq.(A6). The contribution of the electrostatic potential in the LHS is that for cases with $|\nabla s\times\mathbf{B}\cdot\nabla\Phi| \ll 2T_a\left|e_a^{-1}\nabla s\times\mathbf{B}\cdot\nabla\ln B\right| \Leftrightarrow cB^{-2}|\nabla s\times\mathbf{B}\cdot\nabla\Phi| \ll |\mathbf{v}_{\mathrm{d}a}\cdot\nabla s|$ in the DKEs for thermalized particles (the potential is almost a surface-quantity). Analogous to Ref.9, it also is assumed that the inductive electric field, which is retained only for the confirmation of the Onsager symmetric relation of the Ware pinch and the bootstrap current, has only a parallel component $\mathbf{E}^{(\mathbf{A})} = \left\langle \mathbf{B}\cdot\mathbf{E}^{(\mathbf{A})}\right\rangle \mathbf{B}/\left\langle B^2\right\rangle$. Therefore, we shall calculate fundamentally

$$\nabla\cdot\mathbf{Q}_{\|a} = -\nabla\cdot\mathbf{Q}_{\perp a} - e_a n_a\mathbf{u}_{\perp a}\cdot\nabla_\perp\Phi + \frac{m_a}{2}\left(\int v^2 C_a(f_a)\mathrm{d}^3\mathbf{v} - \left\langle \int v^2 C_a(f_a)\mathrm{d}^3\mathbf{v}\right\rangle\right). \quad \text{(A7)}$$

The local collisional energy exchange between the species due to the poloidal/toroidal variations of the lowest Legendre order $\int v^2 C_a(f_a)\mathrm{d}^3\mathbf{v}$ may be important for the P-S diffusion calculation of multi-ion-species plasmas in an extremely collisional condition[17] and the fast ions' slowing down process discussed in Sec.II. By using Eqs.(A1)-(A2), the perpendicular particle and energy fluxes are given by

$$n_a\mathbf{u}_{\perp a} = -\frac{c}{2e_a}\left(\frac{1}{B^2}\nabla(p_{\perp a}+p_{\|a}) + \nabla\frac{p_{\perp a}-p_{\|a}}{B^2}\right)\times\mathbf{B} - n_a\frac{c\nabla\Phi\times\mathbf{B}}{B^2}, \quad \text{(A8)}$$

$$\mathbf{Q}_{\perp a} = -\frac{c}{e_a}\frac{m_a}{2}\left(\frac{1}{B^2}\nabla(r_{\perp a}+r_{\|a}) + \nabla\frac{r_{\perp a}-r_{\|a}}{B^2}\right)\times\mathbf{B} - \left(\frac{5}{2}p_a + \frac{p_{\perp a}-p_{\|a}}{3}\right)\frac{c\nabla\Phi\times\mathbf{B}}{B^2}. \quad \text{(A9)}$$

In these perpendicular fluxes for the purpose of the parallel flux divergences $\nabla\cdot(n_a\mathbf{u}_{\|a})$ and $\nabla\cdot\mathbf{Q}_{\|a}$, the perpendicular friction forces $\mathbf{F}_{\perp a1}$ and $\mathbf{G}_{\perp a}$ (i.e., collision effects against the gyro-motion) are neglected since $e_{\rm f}c^{-1}B/m_{\rm f} \gg 1/\tau_{\rm S}$ for fast ions and $e_a c^{-1}B \gg \left|l_{22}^{aa}\right|/n_a$ for thermalized particles where l_{22}^{aa} is the friction coefficient.[17]. The divergences of Eqs.(A8-A9) given by $\nabla\cdot(H\nabla F\times\mathbf{B}) = \nabla F\times\mathbf{B}\cdot\nabla H - H\frac{4\pi}{c}\mathbf{J}\cdot\nabla F$ for arbitrary scalar quantities $F(\mathbf{x})$, $H(\mathbf{x})$ are

$$\nabla\cdot(n_a\mathbf{u}_{\perp a}) = -\frac{c}{2e_a}\nabla(p_{\perp a}+p_{\|a})\times\mathbf{B}\cdot\nabla\frac{1}{B^2} - c\nabla\Phi\times\mathbf{B}\cdot\nabla\frac{n_a}{B^2} \tag{A10}$$

and

$$\begin{aligned}\nabla\cdot\mathbf{Q}_{\perp a} &= -\frac{c}{e_a}\frac{m_a}{2}\nabla(r_{\perp a}+r_{\|a})\times\mathbf{B}\cdot\nabla\frac{1}{B^2} - c\nabla\Phi\times\mathbf{B}\cdot\nabla\left(\frac{5}{2}\frac{p_a}{B^2}+\frac{p_{\perp a}-p_{\|a}}{3B^2}\right)\\ &= -\frac{c}{e_a}\frac{m_a}{2}\nabla(r_{\perp a}+r_{\|a})\times\mathbf{B}\cdot\nabla\frac{1}{B^2} - c\nabla\Phi\times\mathbf{B}\cdot\nabla\left(\frac{3}{2}\frac{p_a}{B^2}+\frac{p_{\perp a}+p_{\|a}}{2B^2}+\frac{p_{\perp a}-p_{\|a}}{2B^2}\right).\end{aligned} \tag{A11}$$

Even though $p_{\perp a}+p_{\|a}$, $(p_{\perp a}-p_{\|a})/B^2$, $r_{\perp a}+r_{\|a}$, and $(r_{\perp a}-r_{\|a})/B^2$ of individual species are not always constant on the flux-surfaces $s=$ const, and the $\mathbf{J}_\perp$ vector also sometimes may deviate from the surfaces (i.e., $\mathbf{J}\cdot\nabla s\neq 0$), the $H\mathbf{J}\cdot\nabla F$ in $\nabla\cdot(H\nabla F\times\mathbf{B})$ was neglected by a low-beta approximation. On the other hand, the second term in $\nabla\cdot\mathbf{Q}_{\|a}$ in Eq.(A7) is

$$n_a\mathbf{u}_{\perp a}\cdot\nabla_\perp\Phi = \frac{c}{2e_a}\nabla\Phi\times\mathbf{B}\cdot\left(\frac{1}{B^2}\nabla(p_{\perp a}+p_{\|a})+\nabla\frac{p_{\perp a}-p_{\|a}}{B^2}\right) \tag{A12}$$

because of Eq.(8). Therefore, the first and the second terms in Eq.(A7) become

$$\begin{aligned}&-\nabla\cdot\mathbf{Q}_{\perp a} - e_a n_a\mathbf{u}_{\perp a}\cdot\nabla_\perp\Phi\\ &= \frac{c}{e_a}\frac{m_a}{2}\nabla(r_{\perp a}+r_{\|a})\times\mathbf{B}\cdot\nabla\frac{1}{B^2} + c\nabla\Phi\times\mathbf{B}\cdot\left(\frac{3}{2}\nabla\frac{p_a}{B^2}+\frac{p_{\perp a}+p_{\|a}}{2}\nabla\frac{1}{B^2}\right).\end{aligned} \tag{A13}$$

As already noted, we can retain only components in these equations that satisfy the solubility conditions $\left\langle\nabla\cdot(n_a\mathbf{u}_{\|a})\right\rangle = \left\langle\mathbf{B}\cdot\nabla\left(n_a u_{\|a}/B\right)\right\rangle = 0$ and $\left\langle\nabla\cdot\mathbf{Q}_{\|a}\right\rangle = \left\langle\mathbf{B}\cdot\nabla\left(Q_{\|a}/B\right)\right\rangle = 0$ for the steady-state conditions with $\partial n_a/\partial t = 0$ and $\partial p_a/\partial t = 0$. Components corresponding to $\partial\left\langle n_a\mathbf{u}_a\cdot\nabla V\right\rangle/\partial V$ and $\partial\left\langle\mathbf{Q}_a\cdot\nabla V\right\rangle/\partial V + e_a\left\langle n_a\mathbf{u}_a\cdot\nabla\Phi\right\rangle$ must be removed. For thermalized particles, this is one reason for which we cannot retain the full terms in Eq.(6) that reproduces the electric field term in Eq.(A13). The differential operator $\mathbf{v}_{\rm da}\cdot\nabla$ with the perpendicular guiding center velocity $\mathbf{v}_{\rm da} = e_a^{-1}c\left(m_a v_\|^2/B+\mu\right)\mathbf{b}\times\nabla\ln B$ also cannot be used for the full part of the velocity distribution $f_a(\mathbf{x},\mathbf{v})$ because of this solubility condition. When the electrostatic potential is a constant on the surfaces, such parallel flux divergences guaranteeing the solubility condition by the theorem $\langle\nabla s\times\mathbf{B}\cdot\nabla F\rangle = 0$ for arbitrary scalar quantity $F(\mathbf{x})$ are

$$\mathbf{B}\cdot\nabla\frac{n_a u_{\|a}}{B} = \frac{c}{2e_a}\frac{\partial\left\langle p_{\perp a}+p_{\|a}\right\rangle}{\partial s}\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2} + c\frac{\partial\Phi}{\partial s}\nabla s\times\mathbf{B}\cdot\nabla\frac{n_a}{B^2} \tag{A14}$$

and

$$\mathbf{B}\cdot\nabla\frac{Q_{\|a}}{B}=\frac{c}{e_a}\frac{m_a}{2}\frac{\partial\left\langle r_{\perp a}+r_{\|a}\right\rangle}{\partial s}\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2}+c\frac{\partial\Phi}{\partial s}\nabla s\times\mathbf{B}\cdot\left(\frac{3}{2}\nabla\frac{p_a}{B^2}+\frac{\left\langle p_{\perp a}+p_{\|a}\right\rangle}{2}\nabla\frac{1}{B^2}\right)$$
$$+\frac{m_a}{2}\left(\int v^2C_a(f_a)\mathrm{d}^3\mathbf{v}-\left\langle\int v^2C_a(f_a)\mathrm{d}^3\mathbf{v}\right\rangle\right). \tag{A15}$$

For the diamagnetic flux divergences, this approximation is due to a relation between the radial gradient scale lengths $\left|\frac{\partial}{\partial s}\ln\left\langle p_{\perp a}+p_{\|a}\right\rangle\right|$, $\left|\frac{\partial}{\partial s}\ln\left\langle r_{\perp a}+r_{\|a}\right\rangle\right|\gg\left|\frac{\partial}{\partial s}\ln\left\langle B^{-2}\right\rangle\right|$. It also should be noted that the poloidal/toroidal variations $\left|(p_{\perp a}+p_{\|a})/\left\langle p_{\perp a}+p_{\|a}\right\rangle-1\right|$ and $\left|(r_{\perp a}+r_{\|a})/\left\langle r_{\perp a}+r_{\|a}\right\rangle-1\right|$ of individual species in stellarator/heliotron plasmas are not always the first order of $\rho_a/L_r\propto\langle B\rangle^{-1}$ but often can become $\sim\{(B_{\mathrm{M}}-B_{\min})/(B_{\mathrm{M}}+B_{\min})\}^{3/2}\propto\langle B\rangle^0$ because of the collisionless detrapping ν regime ripple diffusion of light low-Z species,[27] the resonant viscosity of heavy impurity ions,[5,32] and the P-S diffusions of impure plasmas. In spite of this fact, these variations are not taken into account at least in these diamagnetic flux divergences and the corresponding DKE term $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial f_{a\mathrm{M}}/\partial s$. From the viewpoint of the drift approximation, this neglect corresponds to a relation $|\mathbf{v}_{\mathrm{d}a}\cdot\nabla f_{a1}|\ll\left|\left(V_{\|}+V_E\right)f_{a1}\right|$ for the poloidally and toroidally varying gyro-phase-averaged distributions $\overline{f}_a(\mathbf{x},v,\xi)=f_{a\mathrm{M}}(s,v)+f_{a1}(\mathbf{x},v,\xi)$ in the ambipolar conditions with $\partial\Phi/\partial s\neq0$. Since the "anisotropic-pressure species"in the MHD equilibrium are defined as those with $\left\langle p_{\perp a}-p_{\|a}\right\rangle\left\langle(p_{\perp a}-p_{\|a})/B^2\right\rangle>0$, it also is noteworthy that these divergence terms do not always require rigorous surface-averages of $p_{\perp a}+p_{\|a}$ and $r_{\perp a}+r_{\|a}$ themselves but require the substantial radial gradients of them that are consistent with the perpendicular particle/energy fluxes $n_a\mathbf{u}_{\perp a}$, $\mathbf{Q}_{\perp a}$, and the parallel force balance including the field curvature effect $\mathbf{b}\cdot\nabla\ln B$. When the DKE solution gives $\left\langle p_{\perp a}-p_{\|a}\right\rangle\left\langle(p_{\perp a}-p_{\|a})/B^2\right\rangle<0$ for example, the radial gradients are regarded as those of isotropic-pressure species with $\frac{\partial}{\partial s}\left\langle p_{\perp a}+p_{\|a}\right\rangle=2\frac{\partial}{\partial s}\langle p_a\rangle$, $\frac{\partial}{\partial s}\left\langle r_{\perp a}+r_{\|a}\right\rangle=2\frac{\partial}{\partial s}\langle r_a\rangle$, $\frac{\partial}{\partial s}\left\langle(p_{\perp a}-p_{\|a})/B^2\right\rangle=0$, and $\frac{\partial}{\partial s}\left\langle(r_{\perp a}-r_{\|a})/B^2\right\rangle=0$.

In Eq.(A15), the surface-average of the neglected component of the electric field term in Eq.(A13) can be rewritten by using the definition $\mathbf{B}\cdot\nabla\left(\widetilde{U}/B\right)=(\mathbf{B}\times\nabla s)\cdot\nabla B^{-2}$ of a function[3,9] $\widetilde{U}$ and the parallel ($\mathbf{b}\cdot$) component of Eq.(A1) with $\mathbf{b}\cdot\nabla\cdot(p_a\mathbf{I}+\boldsymbol{\pi}_a)=$

$\frac{1}{2}\mathbf{b}\cdot\left[\nabla(p_{\|a}+p_{\perp a})+B^2\nabla\left\{(p_{\|a}-p_{\perp a})/B^2\right\}\right]$, and the result is

$$\begin{aligned}
&c\frac{\partial\Phi}{\partial s}\left\langle\frac{p_{\perp a}+p_{\|a}}{2}\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2}\right\rangle=-c\frac{\partial\Phi}{\partial s}\left\langle\frac{p_{\perp a}+p_{\|a}}{2}\mathbf{B}\cdot\nabla\frac{\widetilde{U}}{B}\right\rangle\\
&=c\frac{\partial\Phi}{\partial s}\left\langle\widetilde{U}\mathbf{b}\cdot\nabla\frac{p_{\perp a}+p_{\|a}}{2}\right\rangle=c\frac{\partial\Phi}{\partial s}\left\{\left\langle(p_{\|a}-p_{\perp a})\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\widetilde{U}\mathbf{b}\right)\cdot\nabla\ln B\right\rangle+\left\langle\widetilde{U}F_{\|a1}\right\rangle\right\}.
\end{aligned}\tag{A16}$$

Because of a relation $\left\langle B\widetilde{U}\right\rangle=0$ as a part of the definition of $\widetilde{U}$, and the aforementioned assumption of the inductive electric field, $\left\langle n_aE_{\|}^{(\mathbf{A})}\widetilde{U}\right\rangle=0$ is used. Therefore the neglected component of $p_{\perp a}+p_{\|a}$ corresponds to $\left\langle\left(\boldsymbol{\Gamma}_a^{\mathrm{bn}}+\boldsymbol{\Gamma}_a^{\mathrm{PS}}\right)\cdot\nabla V\right\rangle\partial\Phi/\partial V$ in Eq.(A6). Analogously, by using the Gauss' theorem[9]

$$\begin{aligned}
\left\langle\nabla\cdot(H\nabla F\times\mathbf{B})\right\rangle&=\frac{\partial}{\partial V}\left\langle H\nabla F\times\mathbf{B}\cdot\nabla V\right\rangle\\
=-\frac{\partial}{\partial V}\left\langle H\nabla V\times\mathbf{B}\cdot\nabla F\right\rangle&=\frac{\partial}{\partial V}\left\langle F\nabla V\times\mathbf{B}\cdot\nabla H\right\rangle,
\end{aligned}\tag{A17}$$

we can confirm the fact that the surface-averages of the diamagnetic terms in Eqs.(A10-A11) that are removed in Eqs.(A14-A15) correspond to $\partial\left\langle\left(\boldsymbol{\Gamma}_a^{\mathrm{bn}}+\boldsymbol{\Gamma}_a^{\mathrm{PS}}\right)\cdot\nabla V\right\rangle/\partial V$ and $\partial\left\langle\left(\mathbf{Q}_a^{\mathrm{bn}}+\mathbf{Q}_a^{\mathrm{PS}}\right)\cdot\nabla V\right\rangle/\partial V$. Further linearization for the $\nabla\Phi\times\mathbf{B}\cdot\nabla$ terms by

$$\begin{aligned}
\nabla s\times\mathbf{B}\cdot\nabla\frac{n_a}{B^2}&\cong\langle n_a\rangle\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2}+\left\langle B^2\right\rangle^{-1}\nabla s\times\mathbf{B}\cdot\nabla n_a,\\
\nabla s\times\mathbf{B}\cdot\nabla\frac{p_a}{B^2}&\cong\langle p_a\rangle\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2}+\left\langle B^2\right\rangle^{-1}\nabla s\times\mathbf{B}\cdot\nabla p_a
\end{aligned}$$

corresponds to the approximation of the $\mathbf{E}\times\mathbf{B}$ effect in the DKE using V_E^{DKES} for the solubility condition and the anti-symmetric property $\left\langle\int\widehat{g}_a\left(V_E^{\mathrm{DKES}}\widehat{f}_af_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}\right\rangle=-\left\langle\int\widehat{f}_a\left(V_E^{\mathrm{DKES}}\widehat{g}_af_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}\right\rangle$.

On the radial gradients in Eqs.(A14-A15), the following facts also should be considered. The negative value $\left\langle p_{\perp a}-p_{\|a}\right\rangle\left\langle(p_{\perp a}-p_{\|a})/B^2\right\rangle\le 0$ given by the usual surface-averaging $\langle F\rangle\equiv\oint\oint F\sqrt{g}\mathrm{d}\theta\mathrm{d}\zeta/\oint\oint\sqrt{g}\mathrm{d}\theta\mathrm{d}\zeta$ for actual geometrical shapes of the flux-surfaces corresponds to a condition where we can regard the gradients as those of isotropic-pressure species for all of the force balances in the MHD equilibrium, the classical diffusions, and the DKE for determining the gyro-phase-averaged distribution. The radial gradient term $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial\left\langle\overline{f}_a\right\rangle/\partial s$ in the DKE also can be regarded as those of isotropic velocity distribution $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial f_{a\mathrm{M}}/\partial s$ when its solution does not satisfy $\left\langle p_{\perp a}-p_{\|a}\right\rangle\left\langle(p_{\perp a}-p_{\|a})/B^2\right\rangle>0$ and $\left\langle r_{\perp a}-r_{\|a}\right\rangle\left\langle(r_{\perp a}-r_{\|a})/B^2\right\rangle>0$ in the parallel force balance including $\mathbf{b}\cdot\nabla\ln B$. Since the geometrical shapes of the surfaces are not essential for the DKE described by using the flux-surface coordinates, however, this judgment does not always require the usual surface-averaging. The required inputs for the DKE from the equilibrium configuration calculation are only some surface-quantities $(\chi', \psi', B_\zeta, B_\theta)$ and the field strength $B\,(s, \theta, \zeta)$ in the contravariant and the covariant expressions of the $\mathbf{B}$-field.[3,9,27] Based on this fact, one insistence of Ref.3 is that results for quasi-symmetric fields and geometrically symmetric fields must be identical. In this kind of theory for the gyro-phase-averaged velocity distributions, the appearance of the $\langle\cdot\rangle$ in various derivation steps in various formulas usually corresponds to the use of the theorems $\langle H\mathbf{B}\cdot\nabla F\rangle = -\langle F\mathbf{B}\cdot\nabla H\rangle$, or $\langle H\nabla s\times\mathbf{B}\cdot\nabla F\rangle = -\langle F\nabla s\times\mathbf{B}\cdot\nabla H\rangle$, or $\langle\nabla\cdot\mathbf{F}_\perp\rangle = \partial\langle\mathbf{F}_\perp\cdot\nabla V\rangle/\partial V$. For example, the solving procedure in Sec.II is based on a theorem $\langle H\mathbf{B}\cdot\nabla F\rangle = -\langle F\mathbf{B}\cdot\nabla H\rangle$. However, to judge whether the radial gradient term in the DKE or corresponding diamagnetic flux divergences in Eqs.(A14-A15) can be regarded as those of isotropic-pressure species is irrelative to these theorems. Furthermore, some solving procedures for the DKE essentially require Fourier expansions in the Boozer coordinates system with the Jacobian $\sqrt{g_{\mathrm{B}}} = (V'/4\pi^2)\langle B^2\rangle/B^2$. In the P-S diffusion calculation, for example, linear algebraic equations of the Fourier-Laguerre expansion coefficients are used for including the field particle portion $C_{ab}\,(f_{a\mathrm{M}}, f_{b1})$, and this procedure is possible for non-symmetric stellarator/heliotron plasmas with finite radial electric fields $\partial\Phi/\partial s \neq 0$ only when the Boozer coordinates system is adopted. In addition to this example, it is known from past experiences that this coordinates system is suited for Fourier series expressions of various quantities in stellarator/heliotron plasmas rather than other coordinates systems.[35] For these DKEs described using the Boozer coordinates, we can use also a simple plane-averaging $\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F\mathrm{d}\theta_{\mathrm{B}}\mathrm{d}\zeta_{\mathrm{B}} = \langle B^2F\rangle/\langle B^2\rangle$ for the $(\theta_{\mathrm{B}}, \zeta_{\mathrm{B}})$ space for this judgment on the characteristic of $p_{\perp a} + p_{\|a}$, $(p_{\perp a} - p_{\|a})/B^2$, $r_{\perp a} + r_{\|a}$, and $(r_{\perp a} - r_{\|a})/B^2$. As a result, the radial gradient term $(\mathbf{v}_{\mathrm{d}a}\cdot\nabla s)\,\partial\left\langle\overline{f}_a\right\rangle/\partial s$ can be regarded as that of isotropic-pressure species even when $\left\langle B^2(p_{\perp a} - p_{\|a})\right\rangle\left\langle p_{\perp a} - p_{\|a}\right\rangle < 0$ and $\left\langle B^2(r_{\perp a} - r_{\|a})\right\rangle\left\langle r_{\perp a} - r_{\|a}\right\rangle < 0$. For the perpendicular particle/energy fluxes and the resultant classical diffusions in this situation, however, it is better to include $\langle B\rangle\frac{\partial}{\partial s}\left\langle\int v^k P_2(\xi) f_a \mathrm{d}^3\mathbf{v}/B^3\right\rangle$ with $k = 2, 4$ in $(\nabla\cdot\boldsymbol{\pi}_a)_\perp/B^2$ and $\{\nabla\cdot(\mathbf{r}_a - r_a\mathbf{I})\}_\perp/B^2$.

Appendix B: The Braginskii's matrix elements for the anisotropy relaxation

By using the orthogonal relation

$$\frac{4}{\sqrt{\pi}}\int_0^\infty \exp(-x^2)x^6 L_j^{(5/2)}(x^2)L_k^{(5/2)}(x^2)\mathrm{d}x = \begin{cases} 0 \ (j\neq k) \\ \dfrac{\Gamma(1/2+j+3)}{j!} = \dfrac{(2j+5)!!}{2^{j+2}j!} \ (j=k) \end{cases} \tag{B1}$$

of the Sonine polynomials $L_j^{(5/2)}(x^2)$, the polynomial expansion of the second Legendre order $l=2$ in the velocity distribution is defined by

$$\begin{aligned} f_a^{(l=2)} &= \frac{m_a v^2}{3\left\langle T_a\right\rangle} P_2(\xi)\left\langle f_{a\mathrm{M}}\right\rangle \sum_{j=0}^{\infty} p_{2aj} L_j^{(5/2)}(x_a^2), \\ p_{2aj} &\equiv \frac{15\cdot 2^j j!}{(2j+5)!!}\frac{m_a}{\left\langle p_a\right\rangle}\int v^2 P_2(\xi) L_j^{(5/2)}(x_a^2) f_a \mathrm{d}^3\mathbf{v}. \end{aligned} \tag{B2}$$

The lowest order term in this series is the pressure anisotropy $p_{2a0} = (p_{\|a} - p_{\perp a})/\left\langle p_a\right\rangle$. There are two methods for obtaining the anisotropy relaxation matrix elements for the algebraic handling of these expansion coefficients p_{2aj}. One method is explained in Ref.36, and the other method is to combine formulas for $\int v^n P_l(\xi) C_{ab}\left(f_a, f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}$, $\int x_a^l P_l(\xi) L_j^{(l+1/2)}(x_a^2) C_{ab}\left(f_{a\mathrm{M}}, f_b\right)\mathrm{d}^3\mathbf{v}$, and $\int_0^\infty x_a^{2n-1}\Phi(x_b)\exp\left(-x_a^2\right)\mathrm{d}x_a$ in Refs.9 and 19. Result for the diagonal part (ions) in Eq.(42) corresponding to $\sum_b C_{ab}\left(f_{a1}, f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}}, f_{a1}\right)$ with $a\neq \mathrm{e}$ are expressed by using the Braginskii's collision time $\tau_{ab} \equiv 3m_a^2 v_{\mathrm{T}a}^3/\left(16\sqrt{\pi} n_b e_a^2 e_b^2 \ln\Lambda_{ab}\right)$ as follows:

$$\begin{aligned} &\int x_a^2 P_2(\xi)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) f_{a\mathrm{M}}\right)\right]\mathrm{d}^3\mathbf{v} \\ &= -\frac{3}{10}\sum_{b\neq a,\mathrm{e}}\frac{n_a}{\tau_{ab}}\left(3+5\frac{m_a}{m_b}\right)\left(1+\frac{m_a}{m_b}\right)^{-3/2} - \frac{9}{10\sqrt{2}}\frac{n_a}{\tau_{aa}} - \frac{3}{2}\frac{n_a}{\tau_{a\mathrm{e}}}\frac{v_{\mathrm{T}a}}{v_{\mathrm{Te}}}, \end{aligned}$$

$$\begin{aligned} &\int x_a^2 P_2(\xi)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2) f_{a\mathrm{M}}\right)\right]\mathrm{d}^3\mathbf{v} \\ &= \int x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) f_{a\mathrm{M}}\right)\right]\mathrm{d}^3\mathbf{v} \\ &= -\frac{9}{20}\sum_{b\neq a,\mathrm{e}}\frac{n_a}{\tau_{ab}}\left(3+7\frac{m_a}{m_b}\right)\left(1+\frac{m_a}{m_b}\right)^{-5/2} - \frac{27}{40\sqrt{2}}\frac{n_a}{\tau_{aa}}, \end{aligned}$$

$$\int x_a^2 P_2(\xi)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2) f_{a\mathrm{M}}\right)\right] \mathrm{d}^3\mathbf{v}$$

$$= \int x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) f_{a\mathrm{M}}\right)\right] \mathrm{d}^3\mathbf{v}$$

$$= -\frac{27}{16}\sum_{b\neq a,\mathrm{e}} \frac{n_a}{\tau_{ab}}\left(1+3\frac{m_a}{m_b}\right)\left(1+\frac{m_a}{m_b}\right)^{-7/2} - \frac{27}{64\sqrt{2}}\frac{n_a}{\tau_{aa}},$$

$$\int x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2) f_{a\mathrm{M}}\right)\right] \mathrm{d}^3\mathbf{v}$$

$$= -\frac{3}{10}\sum_{b\neq a,\mathrm{e}} \frac{n_a}{\tau_{ab}}\left\{35\left(\frac{m_a}{m_b}\right)^3 + \frac{77}{2}\left(\frac{m_a}{m_b}\right)^2 + \frac{185}{4}\frac{m_a}{m_b} + \frac{51}{4}\right\}\left(1+\frac{m_a}{m_b}\right)^{-7/2} - \frac{123}{32\sqrt{2}}\frac{n_a}{\tau_{aa}} - \frac{21}{2}\frac{n_a}{\tau_{a\mathrm{e}}}\frac{v_{\mathrm{T}a}}{v_{\mathrm{Te}}},$$

$$\int x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2) f_{a\mathrm{M}}\right)\right] \mathrm{d}^3\mathbf{v}$$

$$= \int x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) L_1^{(5/2)}(x_a^2) f_{a\mathrm{M}}\right)\right] \mathrm{d}^3\mathbf{v}$$

$$= -\frac{27}{20}\sum_{b\neq a,\mathrm{e}} \frac{n_a}{\tau_{ab}}\left\{21\left(\frac{m_a}{m_b}\right)^3 + \frac{33}{2}\left(\frac{m_a}{m_b}\right)^2 + \frac{135}{8}\frac{m_a}{m_b} + \frac{31}{8}\right\}\left(1+\frac{m_a}{m_b}\right)^{-9/2} - \frac{4401}{1280\sqrt{2}}\frac{n_a}{\tau_{aa}},$$

$$\int x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2)\left[\sum_b C_{ab}\left(x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2) f_{a\mathrm{M}},\ f_{b\mathrm{M}}\right) + C_{aa}\left(f_{a\mathrm{M}},\ x_a^2 P_2(\xi) L_2^{(5/2)}(x_a^2) f_{a\mathrm{M}}\right)\right] \mathrm{d}^3\mathbf{v}$$

$$= -\frac{27}{80}\sum_{b\neq a,\mathrm{e}} \frac{n_a}{\tau_{ab}}\left\{105\left(\frac{m_a}{m_b}\right)^5 + 189\left(\frac{m_a}{m_b}\right)^4 + 462\left(\frac{m_a}{m_b}\right)^3\right.$$

$$\left. +277\left(\frac{m_a}{m_b}\right)^2 + \frac{1317}{8}\frac{m_a}{m_b} + \frac{235}{8}\right\}\left(1+\frac{m_a}{m_b}\right)^{-11/2} - \frac{107001}{10240\sqrt{2}}\frac{n_a}{\tau_{aa}} - \frac{567}{16}\frac{n_a}{\tau_{a\mathrm{e}}}\frac{v_{\mathrm{T}a}}{v_{\mathrm{Te}}}.$$

When they are used for electrons $a = \mathrm{e}$, $\sum_{b\neq a,\mathrm{e}}$ is replaced by $\sum_{b\neq \mathrm{e}}$ with $m_\mathrm{e}/m_b = 0$, and $v_{\mathrm{T}a}/v_{\mathrm{Te}} = v_{\mathrm{Te}}/v_{\mathrm{Te}}$ must be omitted. The matrix elements for the field particle portion corresponding to the non-diagonal parts in Eq.(42) are

$$\int x_a^2 P_2(\xi) C_{ab}\left(f_{a\mathrm{M}},\ x_b^2 P_2(\xi) f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} = \frac{3n_a}{5\tau_{ab}}\frac{m_a}{m_b}\left(1+\frac{m_a}{m_b}\right)^{-3/2}$$

$$= \int x_b^2 P_2(\xi) C_{ba}\left(f_{b\mathrm{M}},\ x_a^2 P_2(\xi) f_{a\mathrm{M}}\right) \mathrm{d}^3\mathbf{v},$$

$$\int x_a^2 L_1^{(5/2)}(x_a^2) P_2(\xi) C_{ab}\left(f_{a\mathrm{M}},\ x_b^2 P_2(\xi) f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} = \frac{9 n_a}{5\tau_{ab}} \frac{m_a}{m_b} \left(1 + \frac{m_a}{m_b}\right)^{-5/2}$$
$$= \int x_b^2 P_2(\xi) C_{ba}\left(f_{b\mathrm{M}},\ x_a^2 L_1^{(5/2)}(x_a^2) P_2(\xi) f_{a\mathrm{M}}\right) \mathrm{d}^3\mathbf{v},$$

$$\int x_a^2 L_1^{(5/2)}(x_a^2) P_2(\xi) C_{ab}\left(f_{a\mathrm{M}},\ x_b^2 L_1^{(5/2)}(x_b^2) P_2(\xi) f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} = \frac{9 n_a}{\tau_{ab}} \left(\frac{m_a}{m_b}\right)^2 \left(1 + \frac{m_a}{m_b}\right)^{-7/2}$$
$$= \int x_b^2 L_1^{(5/2)}(x_b^2) P_2(\xi) C_{ba}\left(f_{b\mathrm{M}},\ x_a^2 L_1^{(5/2)}(x_a^2) P_2(\xi) f_{a\mathrm{M}}\right) \mathrm{d}^3\mathbf{v},$$

$$\int x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) C_{ab}\left(f_{a\mathrm{M}},\ x_b^2 P_2(\xi) f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} = \frac{27}{8} \frac{n_a}{\tau_{ab}} \frac{m_a}{m_b} \left(1 + \frac{m_a}{m_b}\right)^{-7/2}$$
$$= \int x_b^2 P_2(\xi) C_{ba}\left(f_{b\mathrm{M}},\ x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) f_{a\mathrm{M}}\right) \mathrm{d}^3\mathbf{v},$$

$$\int x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) C_{ab}\left(f_{a\mathrm{M}},\ x_b^2 L_1^{(5/2)}(x_b^2) P_2(\xi) f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} = \frac{189}{8} \frac{n_a}{\tau_{ab}} \left(\frac{m_a}{m_b}\right)^2 \left(1 + \frac{m_a}{m_b}\right)^{-9/2}$$
$$= \int x_b^2 L_1^{(5/2)}(x_b^2) P_2(\xi) C_{ba}\left(f_{b\mathrm{M}},\ x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) f_{a\mathrm{M}}\right) \mathrm{d}^3\mathbf{v},$$

$$\int x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) C_{ab}\left(f_{a\mathrm{M}},\ x_b^2 L_2^{(5/2)}(x_b^2) P_2(\xi) f_{b\mathrm{M}}\right) \mathrm{d}^3\mathbf{v} = \frac{5103}{64} \frac{n_a}{\tau_{ab}} \left(\frac{m_a}{m_b}\right)^3 \left(1 + \frac{m_a}{m_b}\right)^{-11/2}$$
$$= \int x_b^2 L_2^{(5/2)}(x_b^2) P_2(\xi) C_{ba}\left(f_{b\mathrm{M}},\ x_a^2 L_2^{(5/2)}(x_a^2) P_2(\xi) f_{a\mathrm{M}}\right) \mathrm{d}^3\mathbf{v}.$$

Since $f_{a\mathrm{M}} \propto \exp\left(-x_a^2\right)$ and $x_a \equiv v/\sqrt{2\left\langle T_{\mathrm{i}}\right\rangle/m_a} \equiv v/v_{\mathrm{T}a}$ of ions in this paper are defined by using the averaged common temperature $\langle T_{\mathrm{i}}\rangle \equiv \sum_{a\neq \mathrm{e,f}} \langle p_a\rangle / \sum_{a\neq \mathrm{e,f}} \langle n_a\rangle$ as stated in the introduction, $\left(v_{\mathrm{T}b}/v_{\mathrm{T}a}\right)^2 = m_a/m_b$ is used for the ion velocity ratios in these derivations. As a result, the matrix in Eq.(42) is symmetric because of the self-adjoint property of of the Coulomb collision operator with the linearization using this $f_{a\mathrm{M}}$ with the common temperature.

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